Propositional Logic Refresher

Meaning of Atoms

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How do you call them?

Models for propositional logic are called *valuations*.

Examples

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Some valuation Let $A = \{p, q, r\}$. Then a valuation v_1 might assign p to T, q to F and r to T.

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$p^{v_1} = T$, $q^{v_1} = F$, $r^{v_1} = T$.

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More formally

$p^{v_1} = T, q^{v_1} = F, r^{v_1} = T$.
write $v_1(p)$ instead of p^{v_1}

Building Propositions

We would like to build larger propositions, such as arguments, out of smaller ones, such as propositional atoms. We do this using *operators* that can be applied to propositions, and yield propositions.

Unary Operators

Let p be an atom.

All possibilities

The following options exist:

1. $p^v = F$: $(op(p))^v = F$.

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The fourth operator *negates* its argument, T becomes F and F becomes T. We call this operator *negation*, and write $\neg p$ (pronounced “not p”).

Nullary Operators are Constants

The constant $\top$

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Nullary Operators are Constants

The constant $\top$

The constant $\top$ always evaluates to T, regardless of the valuation.

The constant $\bot$

The constant $\bot$ always evaluates to F, regardless of the valuation.

Binary Operators: 16 choices

p	q	$op_1(p, q)$	$op_2(p, q)$	$op_3(p, q)$	$op_4(p, q)$
F	F	F	F	F	F
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_5(p,q)$	$op_6(p,q)$	$op_7(p,q)$	$op_8(p,q)$
F	F	F	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_9(p, q)$	$op_{10}(p, q)$	$op_{11}(p, q)$	$op_{12}(p, q)$
F	F	T	T	T	T
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_{13}(p, q)$	$op_{14}(p, q)$	$op_{15}(p, q)$	$op_{16}(p, q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Three Famous Ones

op_2 : $op_2(p, q)$ is T when p is T *and* q is T, and F otherwise.

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op_{14} : $op_{14}(p, q)$ is T when p is F *or* q is T, and F otherwise.

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op_{14} : $op_{14}(p, q)$ is T when p is F *or* q is T, and F otherwise. Called *implication*, denoted $p \rightarrow q$.

Inductive Definition

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the least set F that fulfills the following rules:

- The constant symbols $\bot$ and $\top$ are in F.
- Every element of A is in F.
- If ϕ is in F, then $(\neg\phi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \wedge \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \vee \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \rightarrow \psi)$ is also in F.

Example

$$(((\neg p) \land q) \rightarrow (\top \land (q \lor (\neg r))))$$

is a well-formed formula in propositional logic.

More Compact in BNF

$$\phi ::= p \mid \bot \mid \top \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

(Backus Naur Form)

Convention

The negation symbol $\neg$ binds more tightly than $\wedge$ and $\vee$, and $\wedge$ and $\vee$ bind more tightly than $\rightarrow$. Moreover, $\rightarrow$ is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

Convention

The negation symbol $\neg$ binds more tightly than $\wedge$ and $\vee$, and $\wedge$ and $\vee$ bind more tightly than $\rightarrow$. Moreover, $\rightarrow$ is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

Example

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

can be written as

$$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$$

1. Atoms and Propositions
2. Semantics of Propositional Logic
   - Operations on Truth Values
   - Evaluation of Formulas
3. Proof Theory
4. Soundness and Completeness (preview)

Negating Truth Values

Definition

Function $\backslash : \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B	$\backslash B$
F	T
T	F

Conjunction of Truth Values

Definition

Function $\& : \{F, T\} \times \{F, T\} \to \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \& B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction of Truth Values

Definition

Function $\mid : \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \mid B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Implication of Truth Values

Definition

Function $\Rightarrow$: $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \Rightarrow B_2$
F	F	T
F	T	T
T	F	F
T	T	T

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\bot$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \vee \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \rightarrow \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid Formulas

Definition

A formula is called *valid* if it evaluates to T with respect to every possible valuation.

Examples

Example

Is

$$(((\neg p) \wedge q) \rightarrow (\top \wedge (q \vee (\neg r))))$$

valid?

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Example

Find a valid formula that contains the propositional atoms p, q, r and w.

Sequents

Definition

A sequent consists of propositional formulas $\phi_1, \phi_2, \ldots, \phi_n$, called *premises*, where $n \geq 0$, and a propositional formula ψ called *conclusion*. We write a sequent as follows:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

and say "ψ is provable using the premises $\phi_1, \phi_2, \ldots, \phi_n$".

Introducing $\top$

$$\frac{}{\top}\,[\top i]$$

Rules for Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

Rules for Conjunction

$$\frac{\phi \qquad \psi}{\phi \land \psi}\,[\land i]$$

$$\frac{\phi \land \psi}{\phi}\,[\land e_1]$$

Rules for Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\,[\wedge i]$$

$$\frac{\phi \wedge \psi}{\phi}\,[\wedge e_1]$$

$$\frac{\phi \wedge \psi}{\psi}\,[\wedge e_2]$$

Example

$$p \land q, r \vdash q \land r$$

Example

$$p \wedge q, r \vdash q \wedge r$$

Proof (graphical notation):

$$\dfrac{\dfrac{p \wedge q}{q}[\wedge e_2] \qquad r}{q \wedge r}[\wedge i]$$

Example

$$p \land q, r \vdash q \land r$$

Example

$$p \land q, r \vdash q \land r$$

Proof (text-based notation):

1	$(p \land q)$	premise
2	q	$\land$e 1
3	r	premise
4	$q \land r$	$\land$i 2,3

Double Negation Elimination

$$\frac{\neg\neg\phi}{\phi}\,[\neg\neg i]$$

Implication Elimination

$$\frac{\phi \qquad \phi \to \psi}{\psi} [\to e]$$

We would like...

...to be able to prove:

$$p \to q \vdash \neg\neg p \to \neg q$$

A proof should look like this

$$p \rightarrow q \vdash \neg\neg p \rightarrow q$$

1	$p \rightarrow q$	premise
2	$\neg\neg p$	assumption
3	p	$\neg\neg e$ 2
4	q	$\rightarrow e$ 1, 3
5	$\neg\neg p \rightarrow q$	$\rightarrow_i$ 2–4

Implication Introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi}\;[\to i]$$

Rules for Disjunction

$$\frac{\phi}{\phi \lor \psi}[\lor i_1] \qquad \frac{\psi}{\phi \lor \psi}[\lor i_2]$$

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$$\frac{\phi}{\phi \lor \psi}\,[\lor i_1] \qquad \frac{\psi}{\phi \lor \psi}\,[\lor i_2]$$

$$\frac{\phi \lor \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\,[\lor e]$$

Axioms for ⊥ and Negation

$$\frac{\bot}{\phi}\,[\bot e]$$

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$$\frac{\bot}{\phi}\,[\bot e] \qquad \frac{\phi \qquad \neg\phi}{\bot}\,[\neg e]$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\,[\neg i]$$

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ:

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Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ:

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Proof:

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg$e 1,2
4	$\neg\neg\phi$	$\neg$i 2–3

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ:

$$\phi \vdash \neg\neg\phi$$

can be written like an axiom:

$$\frac{\phi}{\neg\neg\phi}\ [\neg\neg i]$$

Law of Excluded Middle

Lemma (LEM)

$$\frac{}{\phi \lor \neg\phi}\,[LEM]$$

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4. Soundness and Completeness (preview)
    - Entailment
    - Soundness and Completeness

Entailment

Definition

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to $\mathtt{T}$, the formula ψ evaluates to $\mathtt{T}$ as well, we say that $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ, written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Soundness and Completeness

Theorem (Soundness of Propositional Logic)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional formulas.
If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

Soundness and Completeness

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Let $\phi_1, \phi_2, \ldots, \phi_n$ *and* ψ *be propositional formulas.*
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Theorem (Completeness of Propositional Logic)

Let $\phi_1, \phi_2, \ldots, \phi_n$ *and* ψ *be propositional formulas.*
If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$*, then* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$*.*