Why is modal logic decidable

CS 112 Fall 2018

Slides by Petros Potikas
Computational problems

Two computational problems:

1. *Model-checking* problem: is a given formula true at a given state at a given Kripke structure

2. *Validity* problem: is a given formula true in all states of all Kripke structures
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- But in ML we have arbitrary nesting of modalities.
- So, this cannot be captured by bounded quantifier alternation.
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- However, this is not enough, as extensions of ML, as computation-tree logic (CTL) while not captured by $\text{FO}^2$.

Decidability of CTL can be explained by tree-model property, which is enjoyed by CTL, but not by $\text{FP}^2$.

Finally, the tree model property leads to automata-based decision procedures.
Taking a closer look at ML, we see that it is a fragment of 2-variable first-order logic $\text{FO}^2$. $\text{FO}^2$ is more tractable than full first-order logic. However, this is not enough, as extensions of ML, as computation-tree logic (CTL) are not captured by $\text{FO}^2$. CTL can be viewed as a fragment of 2-variable fixpoint logic ($\text{FP}^2$).
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Syntax

Definition

(The Basic Modal Language) Let \( P = \{ P_0, P_1, P_2, \ldots \} \) be a set of sentence letters, or atomic propositions. We also include two special propositions \( \top \) and \( \bot \) meaning ‘true’ and ‘false’ respectively. The set of well-formed formulas of modal logic is the smallest set generated by the following grammar:

\[ P_0, P_1, P_2, \ldots \mid \top \mid \bot \mid \neg A \mid A \lor B \mid A \land B \mid A \rightarrow B \mid \Box A \mid \Diamond A \]

Examples

Modal formulas include: \( \Box \bot, P_0 \rightarrow \Diamond (P_1 \land P_2) \).
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Truth conditions:

1. $(M, s) \models P_i$ iff $s \in \pi(P_i)$
2. $(M, s) \models \top$
3. $(M, s) \not\models \bot$
4. $(M, s) \models \neg A$ iff not $(M, s) \models A$
5. $(M, s) \models A \lor B$ iff either $(M, s) \models A$ or, $(M, s) \models B$, or both
6. $(M, s) \models \Box A$ iff for every $t$, s.t. $R(s, t), (M, t) \models A$
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A sentence true at every possible world in every model is said to be *valid*, written $\models A$
Model-checking problem

Theorem

There is an algorithm that, given a finite Kripke structure $M$, a state $s$ of $M$ and a modal formula $\phi$, determines whether $(M, s) \models \phi$ in time $O(||M|| \times |\phi|)$. 
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**Proof.**

Let $\phi_1, \ldots, \phi_m$ be the subformulas of $\phi$ listed in order of length. Thus $\phi_m = \phi$, and if $\phi_i$ is a subformulas of $\phi_j$, then $i < j$. There are at most $|\phi|$ subformulas, so $m \leq |\phi|$. By induction on $k$, we can show that we can label each state $s$ with $\phi_j$ or $\neg \phi_j$, for $j = 1, \ldots, k$, depending on whether or not $\phi_j$ is true in $s$ in time $O(k||M||)$. Only interesting case is $\phi_{k+1} = \Box \phi_j$, $j < k + 1$. By induction hypothesis, we have that each state has already been labeled with $\phi_j$ or $\neg \phi_j$, so we know if node $s$ can be labeled with $\phi_{k+1}$ or not in time $O(||M||)$. 

□
Characterizing the properties of necessity

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Set of valid formulas can be viewed as a characterization of the properties of necessity.

Two approaches:

1. **Proof-theoretic**: all properties of necessity can be formally derived from a short list of basic properties.

2. **Algorithmic**: we study algorithms that recognize properties of necessity and consider their computational complexity.
Properties of necessity

Some basic properties of necessity:

**Theorem**

*For all formulas $\phi, \psi$, and Kripke structures $M$:

1. If $\phi$ is an instance of a propositional tautology, then $M \models \phi$
2. If $M \models \phi$ and $M \models \phi \rightarrow \psi$, then $M \models \psi$
3. $M \models (\Box \phi \land \Box (\phi \rightarrow \psi)) \rightarrow \Box \psi$
4. If $M \models \phi$, then $M \models \Box \phi$*
Characterizing the properties of necessity: Proof-theoretic

Consider the following axiom system $\mathcal{K}$:

- (A1) All tautologies of propositional calculus
- (A2) $(\Box \phi \land \Box (\phi \rightarrow \psi)) \rightarrow \Box \psi$ (Distribution axiom)
- (R1) From $\phi$ and $\phi \rightarrow \psi$ infer $\psi$ (Modus ponens)
- (R2) From $\phi$ infer $\Box \phi$ (Generalization)
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**Theorem (Kripke ’63)**

$\mathcal{K}$ is a sound and complete axiom system.
The above characterization of the properties of necessity is not constructive.

An algorithm that recognizes valid formulas is another characterization.

First step, if a formula is satisfiable, it is also satisfiable in a finite structure of bounded size (bounded-model property). Stronger than the finite-model property, which asserts that if a formula is satisfiable, then it is satisfiable in a finite structure. This implies that formula $\phi$ is valid in all Kripke structures iff $\phi$ is valid in all finite Kripke structures.

Theorem (Fischer, Ladner '79)
If a modal formula $\phi$ is satisfiable, then $\phi$ is satisfiable in a Kripke structure with at most $2^{\|\phi\|}$ states.
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Characterizing the properties of necessity: algorithmically

- From the above Theorem we can get an algorithm (not efficient) for testing validity of a formula $\phi$: construct all Kripke structures with at most $2^{\|\phi\|}$ states and check if the formula is true in every state of each of these structures.
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The “inherent difficulty” of the problem is given by the next theorem:

**Theorem (Ladner ’77)**

The validity problem for modal logic is PSPACE-complete.
Modal logic vs. First-Order Logic

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- Every Kripke structure \( M \) can be viewed as a relational structure \( M^* \) over the vocabulary \( P^* \).
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- Every Kripke structure $M$ can be viewed as a relational structure $M^*$ over the vocabulary $\mathbb{P}^*$.
- Formally, a mapping from a Kripke structure $M = (S, \pi, R)$ to a relational structure $M^*$ over the vocabulary $\mathbb{P}^*$ has:
  1. domain of $M^*$ is $S$.
  2. for each propositional constant $q \in \mathbb{P}$, the interpretation of $q$ in $M^*$ is the set $\pi(q)$.
  3. the interpretation of the binary predicate $R$, is the binary relation $R$. 

Modal logic decidability
Translation of Modal logic to First-Order Logic

A translation from modal formulas into first-order formulas over the vocabulary \( \mathbb{P}^* \), so that for every modal formula \( \phi \) there is corresponding first-order formula \( \phi^* \) with one free variable (ranging over \( S \)):

1. \( q^* = q(x) \) for a propositional constant \( q \)
2. \( (\neg \phi)^* = \neg (\phi^*) \)
3. \( (\phi \land \psi)^* = (\phi^* \land \psi^*) \)
4. \( (\Box \phi)^* = (\forall y (R(x, y) \rightarrow \phi^*(x/y))) \), where \( y \) is a new variable not appearing in \( \phi^* \) and \( \phi^*(x/y) \) is the result of replacing all free occurrences of \( x \) in \( \phi^* \) by \( y \)
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Example

$(\Box \Diamond q)^* = \forall y (R(x, y) \rightarrow \exists z (R(y, z) \land q(z)))$
Theorem (vBenthem ’74,’85)

1. \((M, s) \models \phi \text{ iff } (M^*, V) \models \phi^*(x), \text{ for each assignment } V \text{ s.t. } V(x) = s.\)

2. \(\phi\) is a valid modal formula iff \(\phi^*\) is a valid first-order formula.

\(\phi^*\) is true of exactly the domain elements corresponding to states \(s\) for which \((M, s) \models \phi\)
Translation of Modal logic to First-Order Logic

Is there a paradox?

Modal logic is essentially a first-order logic. Model-checking in first-order logic is PSPACE-complete while in modal logic in linear time.

Validity is robustly undecidable in first-order logic (decidable only by bounding the alternation of quantifiers), while in modal logic is PSPACE-complete.

Carefully examining propositional modal logic, reveals that it is a fragment of $\text{FO}^2$, e.g. $\forall x \forall y (R(x, y) \rightarrow R(y, x))$ is in $\text{FO}^2$, while $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ is not in $\text{FO}^2$.

Two variables suffice to express modal logic formulas, see the above definition, where new variables are introduced only in the last clause:

Example $(\Box \Box q)^* = \forall y (R(x, y) \rightarrow \forall z (R(y, z) \rightarrow q(z)))$. 

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- Carefully examining propositional modal logic, reveals that it is a fragment of 2-variable first-order logic ($\text{FO}^2$), e.g. $\forall x \forall y (R(x, y) \rightarrow R(y, x))$ is in $\text{FO}^2$, while $\forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow R(x, z))$ is not in $\text{FO}^2$. 

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  \[ \forall x \forall y (R(x, y) \to R(y, x)) \text{ is in FO}^2, \text{ while} \]
  \[ \forall x \forall y \forall z (R(x, y) \land R(y, z) \to R(x, z)) \text{ is not in FO}^2. \]
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- Validity is robustly undecidable in first-order logic (decidable only by bounding the alternation of quantifiers), while in modal logic is PSPACE-complete.
- Carefully examining propositional modal logic, reveals that it is a fragment of 2-variable first-order logic (FO²), e.g.
  \[ \forall x \forall y (R(x, y) \rightarrow R(y, x)) \]  is in FO², while
  \[ \forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow R(x, z)) \]  is not in FO².
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\[(\Box \Box q)^* = \forall y (R(x, y) \rightarrow \forall z (R(y, z) \rightarrow q(z))).\]
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But re-using variables we can avoid introducing new variables. Replace the definition of $\phi^*$ by definition $\phi^+$:

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3. $(\phi \land \psi)^+ = (\phi^* \land \psi^+)$
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But re-using variables we can avoid introducing new variables. Replace the definition of $\phi^*$ by definition $\phi^+$:

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$$(\Box \Box q)^+ = \forall y(R(x, y) \rightarrow \forall x(x = y \rightarrow \forall y(R(x, y) \rightarrow \forall x(x = y \rightarrow q(x)))).$$
Translation of Modal logic to First-Order Logic

Theorem

1. \((M, s) \models \phi \iff (M^*, V) \models \phi^+(x),\) for each assignment \(V\) s.t. \(V(x) = s.\)

2. \(\phi\) is a valid modal formula iff \(\phi^+\) is a valid \(FO^2\) formula.
Complexity of FO²

How hard is to evaluate truth of FO² formulas?
How hard is to evaluate truth of $\text{FO}^2$ formulas?

**Theorem (Immerman ’82, Vardi ’95)**

There is an algorithm that, given a relational structure $M$ over a domain $D$, an $\text{FO}^2$-formula $\phi(x, y)$ and an assignment $V : \{x, y\} \rightarrow D$, determines whether $(M, V) \models \phi$ in time $O(||M||^2 \times ||\phi||)$. 
Complexity of FO^2

Historically, Scott in 1962 showed the first decidability result for FO^2, without equality. The full class FO^2 was considered by Mortimer in 1975, who proved decidability by showing that it has the finite model property.
Complexity of $\text{FO}^2$

- Historically, Scott in 1962 showed the first decidability result for $\text{FO}^2$, without equality. The full class $\text{FO}^2$ was considered by Mortimer in 1975, who proved decidability by showing that it has the finite model property.
- But Mortimer’s proof shows bounded-model property.

**Theorem**

If an $\text{FO}^2$-formula $\phi$ is satisfiable, then $\phi$ is satisfiable in a relational structure with at most $2^{\lvert \phi \rvert}$ elements.
Complexity of $\text{FO}^2$

- To check the validity of a $\text{FO}^2$ formula $\phi$, one has to consider only all structures of exponential size.
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- Note, however, that the validity problem for $\text{FO}^2$ is hard for co-NEXPTIME (Führer81) and also complete, while from Theorem 6 modal logic is PSPACE-complete.
Complexity of FO$^2$

- To check the validity of a FO$^2$ formula $\phi$, one has to consider only all structures of exponential size.
- Further, the translation of modal logic to FO$^2$ is linear, so we have Theorem 5.
- Note, however, that the validity problem for FO$^2$ is hard for co-NEXPTIME (Fürer81) and also complete, while from Theorem 6 modal logic is PSPACE-complete.
- The embedding to FO$^2$ does not give a satisfactory explanation of the tractability of modal logic.
Reflexivity

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- Logical properties of necessity are related with the properties of the graph, e.g. veracity is reflexivity

- A Kripke structure $M = (S, \pi, R)$ is said to be reflexive if the relation $R$ is reflexive. Let $M_r$ be the class of all reflexive Kripke structures.
Reflexivity

- In epistemic logic veracity is needed, what is known is true, i.e. \( \Box \phi \rightarrow \phi \)
- Logical properties of necessity are related with the properties of the graph, e.g. veracity is reflexivity
- A Kripke structure \( M = (S, \pi, R) \) is said to be reflexive if the relation \( R \) is reflexive. Let \( M_r \) be the class of all reflexive Kripke structures.
- Axiom \( \mathcal{T} \): \( \Box p \rightarrow p \)
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Theorem

$\mathcal{T}$ is sound and complete for $M_r$. 
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How hard is validity under the assumption of veracity?
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The validity problem for modal logic in $M_r$ is PSPACE-complete.
Reflexivity

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How hard is validity under the assumption of veracity?

**Theorem**

The validity problem for modal logic in \( M_r \) is PSPACE-complete.

**Theorem**

A modal formula \( \phi \) is valid in \( M_r \) iff the FO\(^2\) \( \forall x (R(x, x) \rightarrow \phi^+) \) is valid.
Axiom system S5

What about other properties of necessity?

Consider introspection:


A Kripke structure $M = (S, \pi, R)$ is said to be reflexive, symmetric, transitive if the relation R is reflexive, symmetric, transitive. Let $M_{rst}$ be the class of all reflexive, symmetric and transitive Kripke structures.

Let $S_5$ be the axiom system obtained from $T$ by adding the two rules of introspection.

Theorem 1

$S_5$ is sound and complete for $M_{rst}$.

The validity problem for $S_5$ is NP-complete.

Symmetry can be expressed by FO$_2$, ∀x, y (R(x, y) → R(y, x)), while transitivity cannot ∀x, y, z (R(x, y) ∧ R(y, z) → R(x, z)).
Axiom system S5

What about other properties of necessity? Consider introspection:

1. Positive introspection - “I know what I know”:
   $\Box p \rightarrow \Box \Box p$.

2. Negative introspection - “I know what I don’t know”:
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A Kripke structure $M = (S, \pi, R)$ is said to be reflexive, symmetric, transitive if the relation $R$ is reflexive, symmetric, transitive.

Let $M$ be the class of all reflexive, symmetric and transitive Kripke structures.

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Theorem 1 $S_5$ is sound and complete for $M$.  

The validity problem for $S_5$ is NP-complete.

Symmetry can be expressed by $\forall x, y (R(x, y) \rightarrow R(y, x))$, while transitivity cannot $\forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z))$.

Modal logic decidability
Axiom system S5

What about other properties of necessity? Consider introspection:

1. Positive introspection - “I know what I know”:

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A Kripke structure \( M = (S, \pi, R) \) is said to be reflexive, symmetric, transitive if the relation \( R \) is reflexive, symmetric, transitive.

Let \( M_{\text{rst}} \) be the class of all reflexive, symmetric and transitive Kripke structures.

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Symmetry can be expressed by FO_2, \( \forall x, y (R(x, y) \to R(y, x)) \), while transitivity cannot \( \forall x, y, z (R(x, y) \land R(y, z) \to R(x, z)) \).
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What about other properties of necessity? Consider introspection:

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\[ \]

A Kripke structure \( M = (S, \pi, R) \) is said to be reflexive, symmetric, transitive if the relation \( R \) is reflexive, symmetric, transitive.

Let \( \text{Rst} \) be the class of all reflexive, symmetric and transitive Kripke structures.

Let \( S_5 \) be the axiom system obtained from \( T \) by adding the two rules of introspection.

Theorem 1: \( S_5 \) is sound and complete for \( \text{Rst} \).

The validity problem for \( S_5 \) is NP-complete.

Symmetry can be expressed by FO\(_2\), \( \forall x, y (R(x, y) \rightarrow R(y, x)) \), while transitivity cannot be expressed by FO\(_2\), \( \forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z)) \).
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What about other properties of necessity? Consider introspection:

2. Negative introspection - “I know what I don’t know”:
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What about other properties of necessity? Consider introspection:

1. Positive introspection - “I know what I know”: $\Box p \rightarrow \Box \Box p$.
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Symmetry can be expressed by FO$_2$, $\forall x, y (R(x, y) \rightarrow R(y, x))$, while transitivity cannot $\forall x, y, z (R(x, y) \land R(y, z) \rightarrow R(x, z))$. 

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Let S5 be the axiom system obtained from \( T \) by adding the two rules of introspection.

Theorem

1. **S5 is sound and complete for \( M_{rst} \).**
2. **The validity problem for S5 is NP-complete.**
Axiom system S5

What about other properties of necessity? Consider introspection:

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- Let $S5$ be the axiom system obtained from $T$ by adding the two rules of introspection.

**Theorem**

1. $S5$ is sound and complete for $M_{rst}$.
2. The validity problem for $S5$ is NP-complete.

Symmetry can be expressed by $\text{FO}^2$, $\forall x, y(R(x, y) \rightarrow R(y, x))$, while transitivity cannot $\forall x, y, z(R(x, y) \land R(y, z) \rightarrow R(x, z))$. 
About decidability of modal logic

- The validity in a modal logic is typically decidable. It is very hard to find a modal logic, where validity is undecidable.
- The translation to FO\(^2\) provides a partial explanation why modal logic is decidable.