

Contents of Part III

1. Analogies and disanalogies between space and time
2. Top-down vs bottom-up theories of space
3. Mereotopology

Analogies and Disanalogies between Space and Time

How does time differ from space?

- ▶ **Dimension**

Time has one dimension, space has three.

- ▶ **Direction**

Time is ordered from past to future, space has no such ordering.

- ▶ **Transience**

Time 'flows': successive moments pass from future to present to past; space is not like that.

How is time similar to space?

- ▶ **Extension**

Time and space are both extended.

- ▶ **Structure**

Time has instants/intervals, space has points/regions.

- ▶ **Metricity**

Extended portions of time and space can both be measured.

- ▶ **Occupancy**

Time and space are both occupied: space by matter and objects, time by processes and events.

Formal analogies

There are many **formal analogies** between space and time, e.g.,

TIME	SPACE
instant	point
interval	region
endpoints of interval	boundary of region
event occurs in interval	object located in region
process goes on in interval	matter fills region

Common relations: Overlap, adjacency, containment

Common problems: Discrete vs Continuous; Vagueness, Indeterminacy, Granularity

Top-down vs bottom-up theories of space

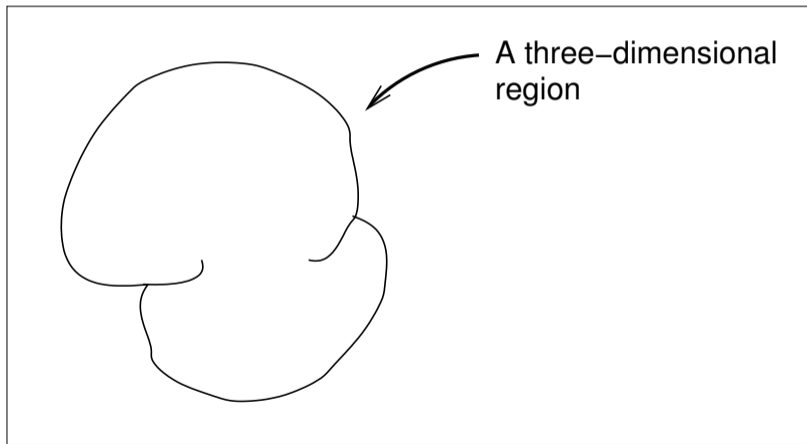
The Container Metaphor for Space

- ▶ We normally locate objects in relation to other objects, e.g.,
 - The key is in the box*
 - The box is on the table*
 - The table is in the bedroom*
 - The bedroom is above the living room*
- ▶ But we also say that objects are located **in space**. We can think of the position of an object as *that portion of space which it exactly fills* (at a given time).
- ▶ *The key is in the box*: the position of the key is a part of the region of space defined by the interior of the box.
- ▶ A widespread approach to formalising statements about location in space is to base it on a theory of spatial points and regions and their properties and interrelations.

Top-down Theory of Space: 3D regions

- ▶ Start with three-dimensional spatial regions: these provide the locations for solid objects.

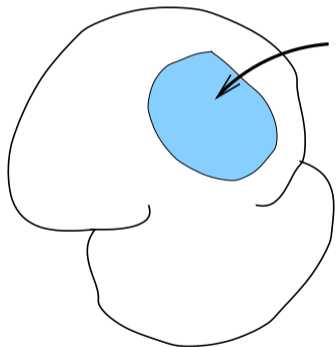
Top-down Theory of Space: 3D regions



Top-down Theory of Space: 2D regions

- ▶ Start with three-dimensional spatial regions: these provide the locations for solid objects.
- ▶ The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.

Top-down Theory of Space: 2D regions

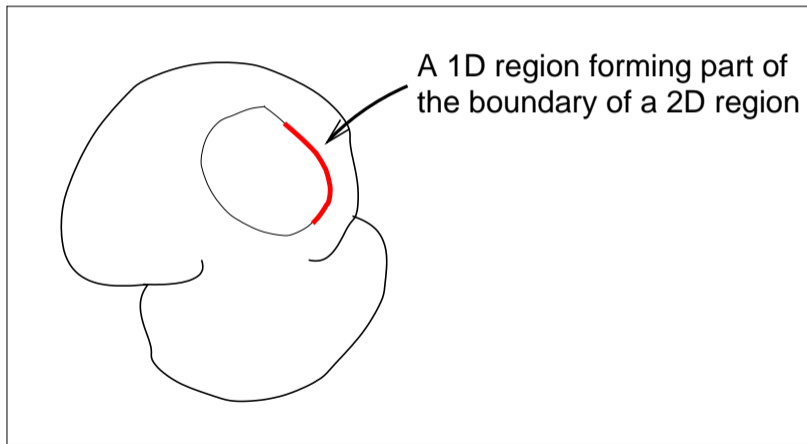


A 2D region forming part of the surface of a 3D region

Top-down Theory of Space: 1D regions

- ▶ Start with three-dimensional spatial regions: these provide the locations for solid objects.
- ▶ The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.
- ▶ The boundary of a 2D region is a one-dimensional region, which can be subdivided into other 1D regions.

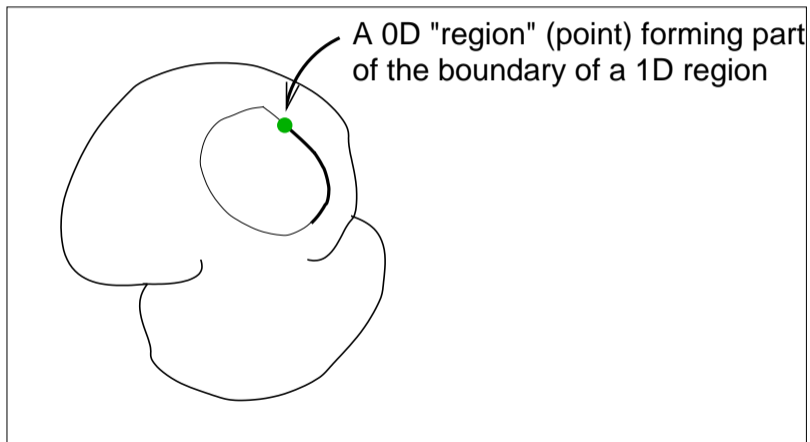
Top-down Theory of Space: 1D regions



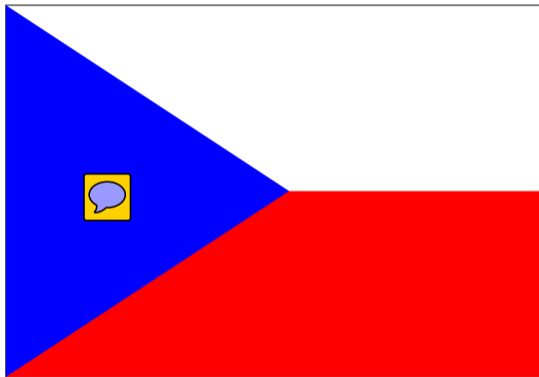
Top-down Theory of Space: 0D “regions” (= points)

- ▶ Start with three-dimensional spatial regions: these provide the locations for solid objects.
- ▶ The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.
- ▶ The boundary of a 2D region is a one-dimensional region, which can be subdivided into other 1D regions.
- ▶ The boundary of a 1D region is a set of points — zero-dimensional regions.

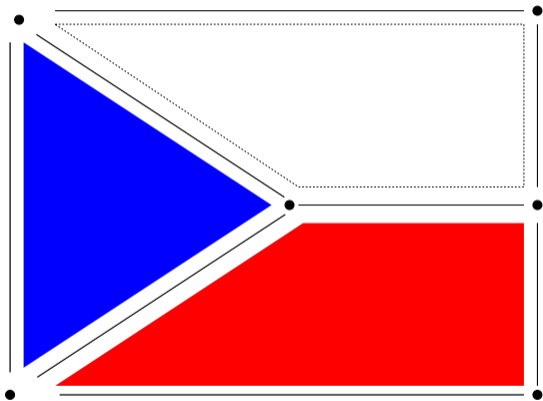
Top-down Theory of Space: 0D regions



Example: The Czech Flag



Example: The Czech Flag dissected



The Czech flag contains three 2D areas, eight 1D line-segments, and six 0D points.

Bottom-up Theory of Space

- ▶ In the bottom-up view, all spatial regions are “made of” points.
- ▶ A point has no size, hence lines, areas, and volumes have to contain infinitely (in fact non-denumerably) many of them. (Does this actually make sense?)
- ▶ On this view, the Czech flag contains infinitely many points, infinitely many lines, and infinitely many areas.
- ▶ But most of them are invisible . . .

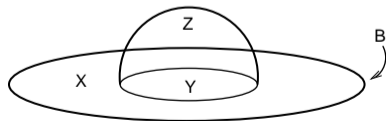
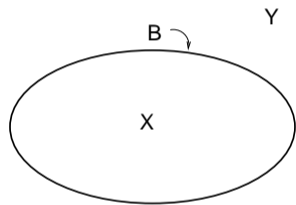
- ▶ If an n -dimensional region is considered as located in an m -dimensional space then it is said to have **co-dimension** $m - n$.
- ▶ Example: In three-dimensional space,
 - ▶ the interior of the earth has co-dimension 0;
 - ▶ the surface of the earth has co-dimension 1;
 - ▶ the coastline of Great Britain has co-dimension 2;
 - ▶ the meeting-point of Germany, France, and Switzerland has co-dimension 3.
- ▶ But if the surface of the earth is taken as the reference space,
 - ▶ the interior of Great Britain has co-dimension 0;
 - ▶ the coastline of Great Britain has co-dimension 1;
 - ▶ the meeting-point of Germany, France, and Switzerland has co-dimension 2.

Spatial relations

The possible spatial relations between regions depends on their dimension and co-dimension.

1. If two regions have co-dimension zero, and have the same boundary, then they are either identical or disjoint. (X and Y are the only regions with boundary B.)

2. But if they have positive co-dimension this is not necessarily the case. (XUY and XUZ both have boundary B.)



In what follows, we shall assume regions of co-dimension 0.

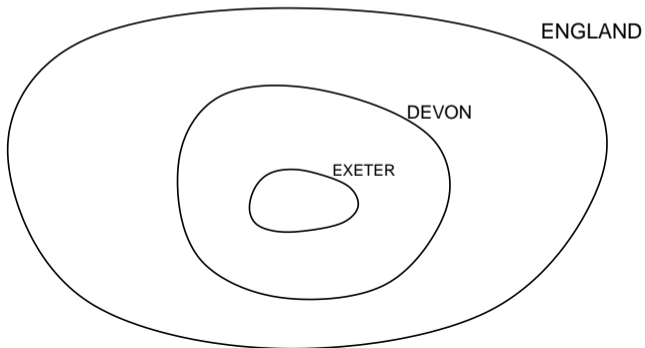
Mereotopology

Exeter is in Devon.

Devon is in England.

What follows?

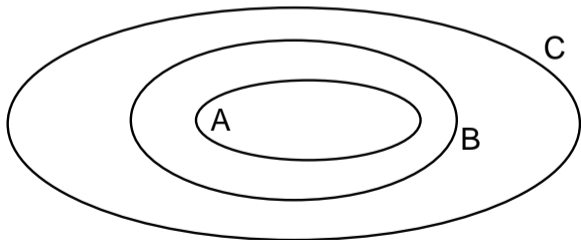
Exeter is in England



A spatial reasoning rule

The Exeter–Devon–England example illustrates a general rule of spatial reasoning:

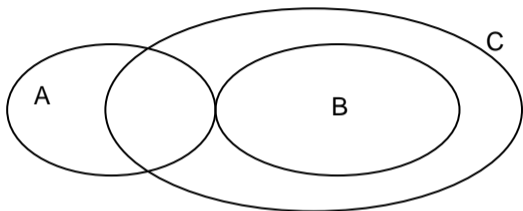
If A is in B and B is in C then A is in C



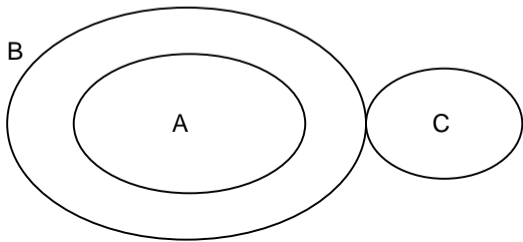
[But: The pain is in my foot; my foot is in my shoe; ...]

Some more rules

If A touches B and B is inside C then A overlaps C .



If A is inside B and B touches C then A is disconnected from C .



Spatial Relationships

The spatial terms in these inferences are

in

inside

touches

overlaps

disconnected

What do we mean by these?

We could characterise them **mathematically**, e.g., in terms of set theory, topology, etc.

Or we could characterise them **logically**, in terms of their relationships to each other and some agreed set of primitives.

Parthood as a primitive

Assume that we have a primitive relation called **parthood**.

Write $P(a, b)$ for 'a is part of b'.

Intuitively, this means that a falls entirely within b . But we need not specify a precise meaning at this stage.

We begin by postulating two axioms for the predicate P :

- ▶ Parthood is **reflexive**:

$$\mathbf{PR} \quad \forall x P(x, x)$$

- ▶ Parthood is **transitive**:

$$\mathbf{PT} \quad \forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(x, z))$$

Defining new relations

We define the relation 'overlap' as follows

DO $O(a, b) =_{\text{def}} \exists x(P(x, a) \wedge P(x, b))$

i.e., 'a overlaps b' means that a and b have a common part.

We can now prove that 'overlap' is **reflexive**:

OR $\forall x O(x, x)$

and **symmetric**:

OS $\forall x \forall y (O(x, y) \rightarrow O(y, x))$

Proof of OR

For any region a we have, by **PR**,

$$P(a, a)$$

and therefore

$$P(a, a) \wedge P(a, a).$$

This implies that

$$\exists x(P(x, a) \wedge P(x, a)).$$

By **DO**, this can be rewritten as $O(a, a)$. Since a was an arbitrary region, we can generalise to

$$\forall xO(x, x),$$

which is **OR**.

More relations

Given regions a and b , either they overlap or they don't.

If they don't overlap, we say they are **disjoint** or **discrete**:

$$DR(a, b) =_{\text{def}} \neg O(a, b)$$

If they do overlap, then there are four possibilities, as follows:

1. a and b are **equal**: $EQ(a, b) =_{\text{def}} P(a, b) \wedge P(b, a)$
2. a is a **proper part** of b : $PP(a, b) =_{\text{def}} P(a, b) \wedge \neg P(b, a)$
3. a has b as a proper part: $PPi(a, b) =_{\text{def}} \neg P(a, b) \wedge P(b, a)$
4. a and b are **partially overlapping**:

$$PO(a, b) =_{\text{def}} \neg P(a, b) \wedge \neg P(b, a) \wedge O(a, b)$$

The system RCC5

The five relations DR, PO, PP, PPI, and EQ form a jointly exhaustive and pairwise disjoint (JEPD) set: any two regions must stand in exactly one of the five relations.

This system is known as **RCC5**.

RCC = **Region Connection Calculus**

(But also, RCC = **Randell, Cui and Cohn**, the authors of the paper in which it was first presented.)

But what they really mean depends on how the primitive P is interpreted.

In particular, does EQ really mean 'equals'?

Does EQ really mean 'equals'?

$EQ(a, b)$ means that a and b are part of each other.

Thus they exactly coincide: does this mean that they must be equal (i.e., the very same object/region/...)?

Only if the following holds:

PA $\forall x \forall y (P(x, y) \wedge P(y, x) \rightarrow x = y)$

(i.e., if the 'part of' relation is **antisymmetric**).

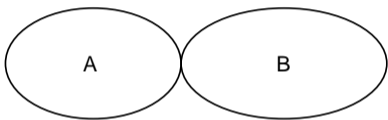
This cannot be derived from our existing axioms. We can postulate it as an extra axiom.

'Touches' and 'Inside'

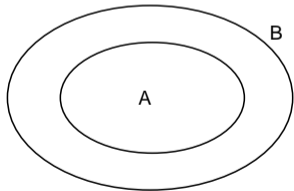
Recall the rule:

If A touches B and B is inside C then A overlaps C .

' A touches B ' means that A and B are disjoint but share part of their boundary.



' A is inside B ' means that A is part of B but does not share any of its boundary.



These relationships cannot be defined in RCC5!

Introducing Connection

The RCC5 relations are called **mereological**: they belong to the theory of parts and wholes.

The relations 'touches' and 'inside' go beyond this, they are **topological**: they belong to the theory of boundaries and connection.

RCC5 can be extended to a theory called RCC8. This is a **mereotopological** theory. It is the spatial analogue of the Interval Calculus.

RCC8 is based on a single primitive relation C .

$C(a, b)$ means that region a is **connected** to region b . This means that they either overlap or touch. (Technically, the greatest lower bound for the distance between a part of a and a part of b is zero.)

Axioms for Connection

The primitive relation C is stipulated to obey the following axioms:

CR $\forall x C(x, x)$

CS $\forall x (C(x, y) \rightarrow C(y, x))$

CE $\forall x \forall y (\forall z (C(x, z) \leftrightarrow C(y, z)) \rightarrow x = y)$

CE says that connection is **extensional**: if x and y are connected to exactly the same things, then they are equal.

Mereology from Topology

Parthood can be defined in terms of connection as follows:

$$\mathbf{DP} \quad P(a, b) =_{\text{def}} \forall x (C(a, x) \rightarrow C(b, x))$$

I.e., a is part of b if b is connected to everything that a is connected to.

(Note: This leads to counter-intuitive results in discrete space!)

Exercise. Given the topological axioms **CR**, **CS**, and **CE**, and the definition **DP**, derive the mereological axioms **PR**, **PT**, and **PA**.

(This means that we don't need to postulate the mereological axioms if we start with connection as our primitive.)

Mereotopological Relations

From C as a primitive, we have already defined P , PP , O , PO and EQ . We now define further relations as follows:

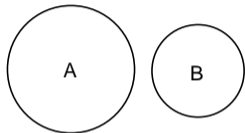
▶ a is **disconnected** from b : $DC(a, b) =_{\text{def}} \neg C(a, b)$

▶ a is **externally connected** to b :
 $EC(a, b) =_{\text{def}} C(a, b) \wedge \neg O(a, b)$

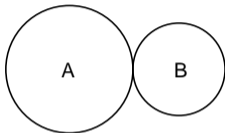
▶ a is a **tangential proper part** of b :
 $TPP(a, b) =_{\text{def}} PP(a, b) \wedge \exists x (EC(x, a) \wedge EC(x, b))$

▶ a is a **non-tangential proper part** of b :
 $NTPP(a, b) =_{\text{def}} PP(a, b) \wedge \neg TPP(a, b)$

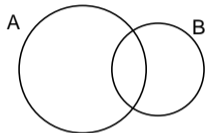
DC(a, b)



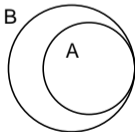
EC(a, b)



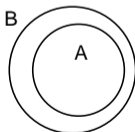
PO(a, b)



TPP(a, b)



NTPP(a, b)



The System RCC8

The inverses of TPP and NTPP are denoted TPPi and NTPPi respectively.

The eight relations DC, EC, PO, EQ, TPP, TPPi, NTPP, and NTPPi form a JEPD set known as **RCC8**.

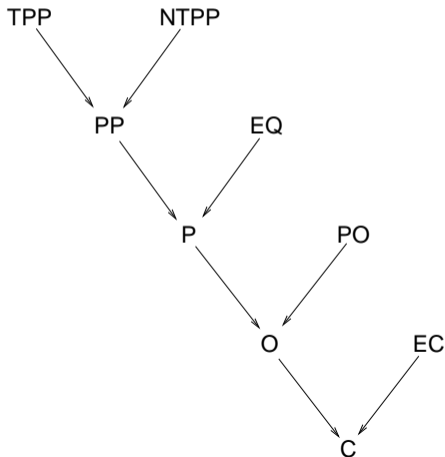
This set forms the basis for many treatments of qualitative spatial reasoning in Artificial Intelligence.

Key reference:

D. A. Randall, Z. Cui, and A. G. Cohn,
'A Spatial Logic Based on Regions and Connection',
in B. Nebel *et al.*, *Principles of Knowledge Representation and Reasoning: Proceedings of the 3rd International Conference (KR'92)*, Morgan Kaufmann (1992), pp. 165–76.

Some Implications in RCC8

These follow immediately from the definitions.



Reasoning with RCC8

Recall the rule:

If A touches B and B is inside C then A overlaps C .

We can write this as: $EC(a, b) \wedge NTPP(b, c) \rightarrow O(a, c)$.

We can prove it as follows:

Assume $EC(a, b)$ and $NTPP(b, c)$.

$NTPP(b, c)$ implies $P(b, c)$, i.e., $\forall x(C(x, b) \rightarrow C(x, c))$. Since we have $C(a, b)$ (implied by $EC(a, b)$), this means that $C(a, c)$.

From $NTPP(b, c)$ we have $\neg\exists x(EC(x, b) \wedge EC(x, c))$. Since we know $EC(a, b)$, this implies $\neg EC(a, c)$.

We now have $C(a, c) \wedge \neg EC(a, c)$, and hence $O(a, c)$, as required.

Some disjunctions

These follow directly from the definitions.

- ▶ $DC(a, b) \vee C(a, b)$
- ▶ $C(a, b) \leftrightarrow EC(a, b) \vee O(a, b)$
- ▶ $O(a, b) \leftrightarrow PO(a, b) \vee P(a, b) \vee P(b, a)$
- ▶ $P(a, b) \leftrightarrow EQ(a, b) \vee PP(a, b)$
- ▶ $PP(a, b) \leftrightarrow TPP(a, b) \vee NTPP(a, b)$

Combining them together, we infer that the RCC8 relations are exhaustive:

$$DC(a, b) \vee EC(a, b) \vee PO(a, b) \vee EQ(a, b) \vee \\ TPP(a, b) \vee NTPP(a, b) \vee TPPI(a, b) \vee NTPPI(a, b)$$

Refining the rule

We've proved $EC(a, b) \wedge NTPP(b, c) \rightarrow O(a, c)$.

Using the disjunction for O, we have

$$EC(a, b) \wedge NTPP(b, c) \rightarrow PO(a, c) \vee P(a, c) \vee P(c, a).$$

Given $P(b, c)$ (from $NTPP(b, c)$), if $P(c, a)$ then $P(b, a)$, so $O(b, a)$, so $\neg EC(b, a)$. But we have $EC(b, a)$ (equivalent to $EC(a, b)$), hence we do not have $P(c, a)$.

Hence we have

$$EC(a, b) \wedge NTPP(b, c) \rightarrow PO(a, c) \vee (P(a, c) \wedge \neg P(c, a)),$$

i.e.,

$$EC(a, b) \wedge NTPP(b, c) \rightarrow PO(a, c) \vee PP(a, c).$$

In terms of pure RCC8 relations,

$$EC(a, b) \wedge NTPP(b, c) \rightarrow PO(a, c) \vee TPP(a, c) \vee NTPP(a, c).$$

This is a special case of the following theorem:

Composition Rules

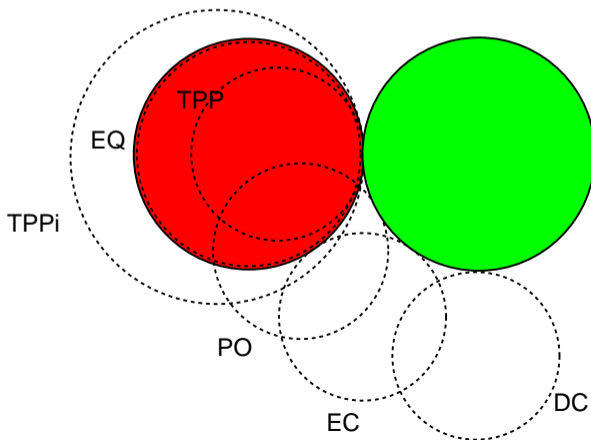
The following table gives the composition rules for all combinations of EC, PO, and TPP. The full table for all eight relations has 64 entries.

	EC	PO	TPP
EC	DC, EC, PO, EQ TPP, TPPi	DC, EC, PO TPP, NTPP	EC, PO TPP, NTPP
PO	DC, EC, PO TPPi, NTPPi	<i>any</i>	PO, TPP NTPP
TPP	DC, EC	DC, EC, PO TPP, NTPP	TPP, NTPP

Illustration: EC with EC

Given $EC(a, b)$ and $EC(b, c)$ the diagram shows the possible relations between a and c .

a is red, b is green, possibilities for c have dotted outlines.



An Example

Suppose a spatial database contains the following assertions:

Fermanagh is a county of Northern Ireland, adjacent to the border.

Northern Ireland is adjacent to the Republic of Ireland.

Leitrim is a county in the interior of the Republic of Ireland.

Fermanagh is adjacent to Leitrim.

Is the database consistent?

Translating into RCC8, we have:

$TPP(f, ni)$

$EC(ni, ri)$

$NTPP(l, ri)$

$EC(f, l)$

Example (contd.)

1. $TPP(f, ni)$
2. $EC(ni, ri)$
3. $NTPP(l, ri)$
4. $EC(f, l)$

Using the composition table, 1 and 2 imply either

5. $DC(f, ri)$

or

6. $EC(f, ri)$.

We can rewrite 3 as

- 3'. $NTPPi(ri, l)$.

Using the table again, both 5 with 3' and 6 with 3' imply

7. $DC(f, l)$

This contradicts 4, so the database is **inconsistent**.

(In fact the statements are all true except 4.)

Summary of RCC8

We have seen that RCC8 is the spatial analogue of the Interval Calculus, and that it has a composition table analogous to the composition table for the Interval Calculus.

Mathematical and computational properties of RCC have been investigated:

1. **Bennett** (1994) showed that RCC8 is decidable.
2. **Renz and Nebel** (1999) showed that the decision problem for RCC8 is NP-complete, and identified a maximal tractable fragment.
3. **Renz** (2002) gave a complete analysis of tractability for RCC8.

What about Conceptual Neighbourhood Diagram for RCC8? This is best handled in the next part when we consider combinations of space and time.