Advanced Logic 2014–15

Dimitri Hendriks

VU University Amsterdam
Theoretical Computer Science

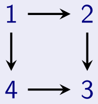
week 2

towards bisimulations

- what can be expressed by the modal language?
- when can two pointed models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ be distinguished by the modal language?
- when should they be viewed as modally identical?
- what is the right semantic equivalence for the basic modal language?

indistinguishable states

Example



- states 2 and 4 cannot be distinguished by a modal formula
- in other words $2 \vDash \varphi$ if and only if $4 \vDash \varphi$, for all formulas φ
- why?

bisimulations

Definition

Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models.
A relation $Z \subseteq W \times W'$ is a **bisimulation** between $\mathcal{M}$ and $\mathcal{M}'$, notation $Z : \mathcal{M} \leftrightarroweq \mathcal{M}'$, if for all pairs $(w, w') \in Z$:

- (**base**) $w \in V(p)$ if and only if $w' \in V'(p)$
- (**zig**) if Rwv then for some $v' \in W'$ we have: $R'w'v'$ and vZv'
- (**zag**) if $R'w'v'$ then for some $v \in W$ we have: Rwv and vZv'

So bisimilar states carry the **same atomic information**, and whenever it is possible to make a transition in one model, it is possible to make a **matching transition** in the other.

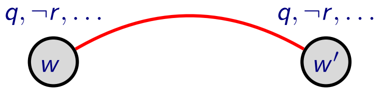
bisimulations: base condition

$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

if $w Z w'$

bisimulations: base condition

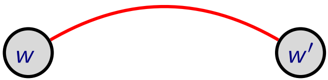


$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

if $w Z w'$
then for all $p \in \text{VAR}$
$w \in V(p)$ if and only if $w' \in V'(p)$

bisimulations: the zig-condition



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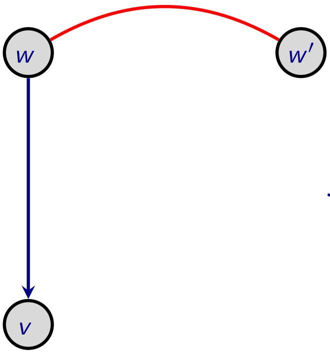
bisimulations: the zig-condition

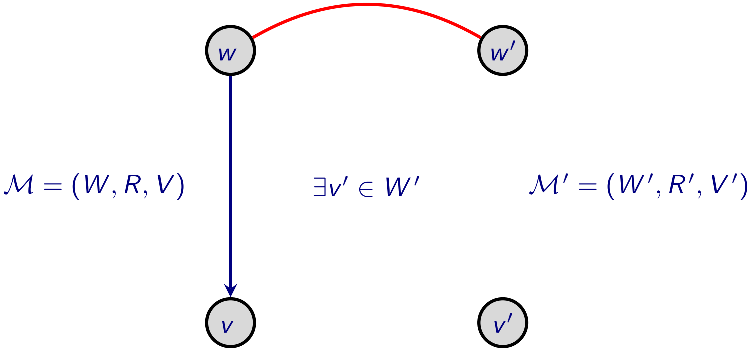


$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

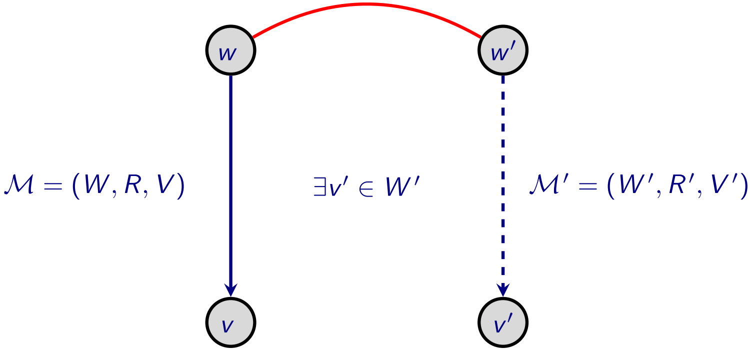
if w Z w' and Rwv

bisimulations: the zig-condition



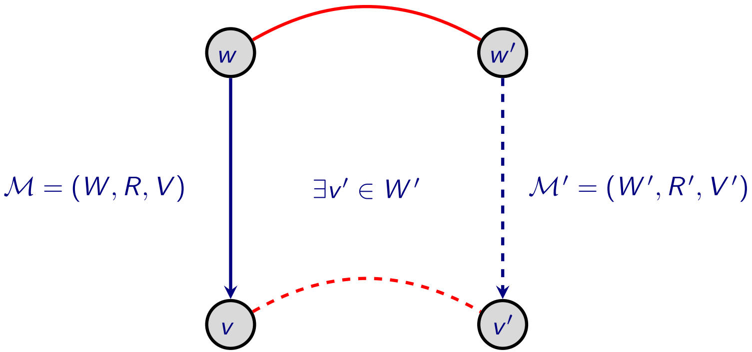
if $w Z w'$ and Rwv
then there exists a point $v' \in W'$

bisimulations: the zig-condition



if $w Z w'$ and Rwv
then there exists a point $v' \in W'$
such that $R'w'v'$

bisimulations: the zig-condition



if wZw' and Rwv
then there exists a point $v' \in W'$
such that $R'w'v'$ and vZv'

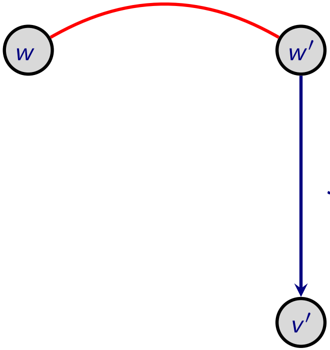
bisimulations: the zag-condition

$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

if $w Z w'$

bisimulations: the zag-condition

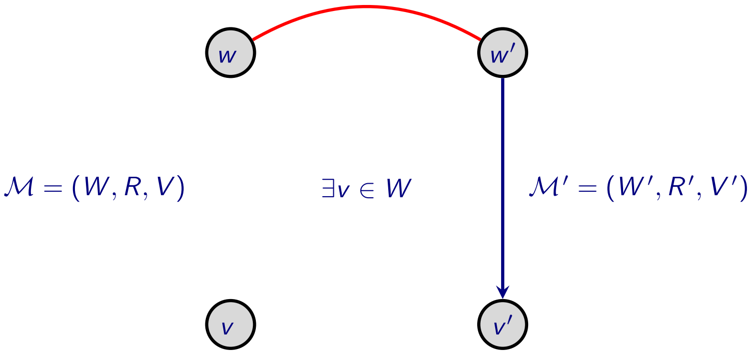


$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

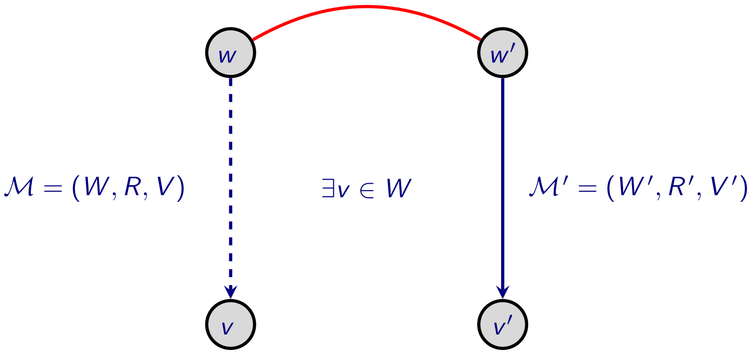
if $w\textcolor{red}{Z}w'$ and $R'w'v'$

bisimulations: the zag-condition



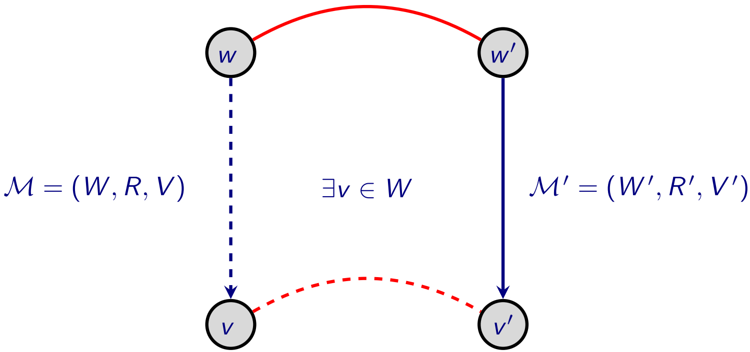
if wZw' and $R'w'v'$
then there exists a point $v \in W$

bisimulations: the zag-condition



if $w \color{red}{Z} w'$ and $R'w'v'$
then there exists a point $v \in W$
such that Rwv

bisimulations: the zag-condition



if wZw' and $R'w'v'$
then there exists a point $v \in W$
such that Rwv and vZv'

bisimilarity

Definition

Two models $\mathcal{M}$ and $\mathcal{M}'$ are **bisimilar**, notation $\mathcal{M} \leftrightarroweq \mathcal{M}'$, if there exists a bisimulation Z such that $Z : \mathcal{M} \leftrightarroweq \mathcal{M}'$.

Two pointed models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are **bisimilar**, notation: $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ or just $w \leftrightarroweq w'$, if $Z : \mathcal{M} \leftrightarroweq \mathcal{M}'$ and wZw' for some bisimulation Z.

bisimilarity

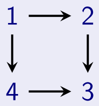
Proposition

$\underline{\leftrightarrow}$ *as a relation between models, is an equivalence relation:*

- $\mathrm{Id} : \mathcal{M} \,\underline{\leftrightarrow}\, \mathcal{M}$ *where* $\mathrm{Id} = \{\, (w, w) \mid w \in W \,\}$.
- *If* $Z : \mathcal{M} \,\underline{\leftrightarrow}\, \mathcal{M}'$, *then* $Z^{-1} : \mathcal{M}' \,\underline{\leftrightarrow}\, \mathcal{M}$, *where* $Z^{-1} = \{\, (w', w) \mid (w, w') \in Z \,\}$.
- *If* $Z_1 : \mathcal{M}_1 \,\underline{\leftrightarrow}\, \mathcal{M}_2$ *and* $Z_2 : \mathcal{M}_2 \,\underline{\leftrightarrow}\, \mathcal{M}_3$ *then* $Z_1 \circ Z_2 : \mathcal{M}_1 \,\underline{\leftrightarrow}\, \mathcal{M}_3$ *where* $Z_1 \circ Z_2 = \{\, (x, z) \mid \exists y \,(\, x Z_1 y \wedge y Z_2 z \,)\,\}$.

example of bisimilar states

Example



States 2 and 4 are bisimilar, since there are bisimulations relating them, for example:

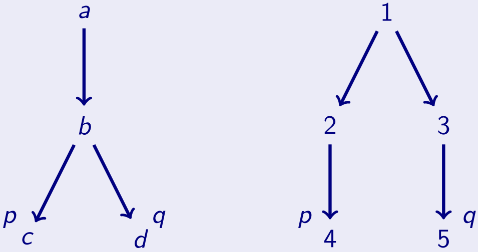
$$
\begin{aligned}
B_1 &= \{(2,4),(3,3)\} \\
B_2 &= \{(1,1),(2,4),(4,2),(3,3)\} \\
B_3 &= \{(1,1),(2,2),(2,4),(3,3),(4,2),(4,4)\}
\end{aligned}
$$

bisimulation games

- players:
  - Spoiler **S** claims (finite) models $\mathcal{M}, s$ and $\mathcal{N}, t$ to be different
  - Duplicator **D** claims they are similar
- play consists of a sequence of links, starting with $s \frown t$
- at each round with current link $m \frown n$:
  - if m and n differ in their atoms, **S** wins
  - if not, **S** has to pick either a successor x of m, or a successor y of n
  - **D** must respond with a matching transition in the other model:
    if **S** took a step $m \to x$ in $\mathcal{M}$, then **D** must find a step $n \to y$ in $\mathcal{N}$
    if **S** took a step $n \to y$ in $\mathcal{N}$, then **D** must find a step $m \to x$ in $\mathcal{M}$
  - play continues with $x \frown y$
  - if a player cannot make a move, (s)he loses
- infinite games (where we return to an already visited link) are won by **D**

example of non-bisimilarity

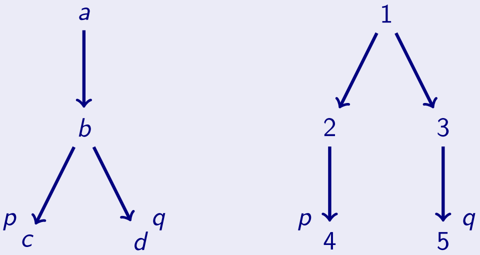
Example



states a and 1 are not bisimilar ...

example of non-bisimilarity

Example

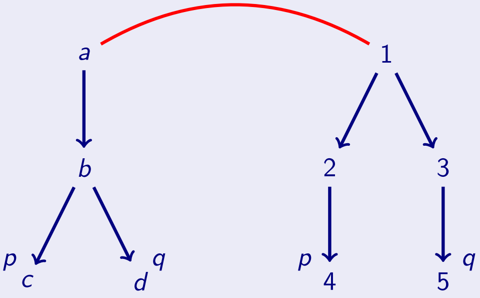


for suppose they were. then $(a, 1) \in Z$ for some bisimulation Z

$$Z \subseteq \{a, b, c, d\} \times \{1, 2, 3, 4, 5\}$$

example of non-bisimilarity

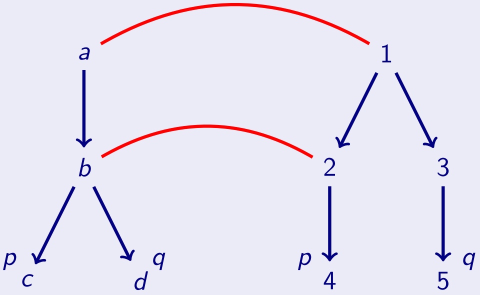
Example



$$Z = \{ (a, 1), \ldots$$

example of non-bisimilarity

Example

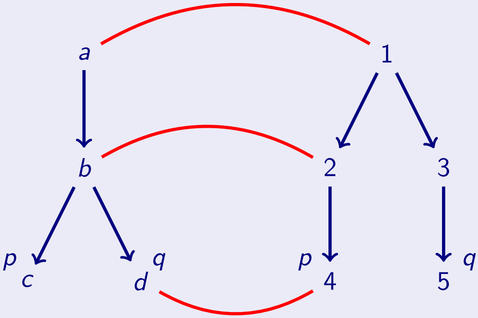


$$Z = \{ (a, 1), (b, 2) \ldots$$

since the step from 1 to 2 has to be matched on the left (zag)

example of non-bisimilarity

Example

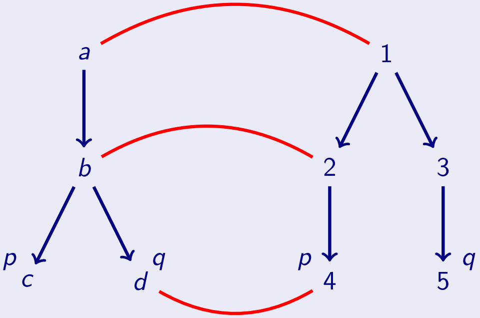


$$Z = \{ (a,1), (b,2), (d,4) \ldots$$

since the step from b to d has to be matched on the right (zig)

example of non-bisimilarity

Example



but d and 4 disagree on their atomic info: $d \not\models p$ whereas $4 \models p$.
hence, there cannot be a bisimulation linking a to 1.

another example of a bisimulation

Example

$$\mathcal{N} = (\mathbb{N}, S) \qquad S = \{(n, n+1) \mid n \in \mathbb{N}\} \qquad V(p) = \{2n \mid n \in \mathbb{N}\}$$

$$\mathcal{F} = (\{e, o\}, R) \qquad R = \{(e, o), (o, e)\} \qquad U(p) = \{e\}$$

State 0 of model $(\mathcal{N}, V)$ bisimulates with state e of model $(\mathcal{F}, U)$.

modal equivalence of states

Definition

Let $\mathcal{M}$ and $\mathcal{M}'$ be models.
A state w of $\mathcal{M}$ and a state w' of $\mathcal{M}'$ are **modally equivalent**, notation $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$, if they satisfy the same formulas:

$$\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w' \quad \text{if and only if} \quad \forall\varphi\,(\mathcal{M}, w \vDash \varphi \text{ iff } \mathcal{M}', w' \vDash \varphi)$$

invariance: $\underline{\leftrightarrow} \subseteq \leftrightsquigarrow$

Theorem

Bisimilar states are modally equivalent:

$$(\mathcal{M}, w \mathrel{\underline{\leftrightarrow}} \mathcal{M}', w') \implies (\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w')$$

In other words: modal truth is **invariant** under bisimulation.

bounded morphisms: functional frame-bisimulations

Definition

Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be frames.
A function $h : W \to W'$ is a **bounded morphism** if it satisfies

- for all $w, v \in W$, if Rwv then $R'h(w)h(v)$
- for all $w \in W$, $v' \in W'$, if $R'h(w)v'$ then there exists $v \in W$ such that $h(v) = v'$ and Rwv

We write $h : \mathcal{F} \twoheadrightarrow \mathcal{F}'$ if h is a **surjective** bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$ (so when the image of h is the entire domain of $\mathcal{F}'$).
We write $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ if $h : \mathcal{F} \twoheadrightarrow \mathcal{F}'$ for some h, and call $\mathcal{F}'$ a **bounded morphic image** of $\mathcal{F}$.

Note that the relation $H = \{(x, h(x)) \mid x \in W\}$ satisfies the zig and zag conditions of bisimulation.

surjective bounded morphisms preserve frame validity

Theorem

A bounded morphic image $\mathcal{F}'$ of $\mathcal{F}$ contains the theory of $\mathcal{F}$, i.e.,

$$(\mathcal{F} \twoheadrightarrow \mathcal{F}') \implies (\mathcal{F} \vDash \varphi \implies \mathcal{F}' \vDash \varphi)$$

application: asymmetry not modally definable

Example

There is no modal formula that characterizes asymmetry ($Rxy \to \neg Ryx$); proof using frames $\mathcal{N}$ en $\mathcal{F}$ from slide 45:

- suppose there was such a formula φ
- then $\mathcal{N} \vDash \varphi$
- let h be defined by $h(2n) = e$ and $h(2n+1) = o$
- then $h : \mathcal{N} \twoheadrightarrow \mathcal{F}$ and so $\mathcal{F} \vDash \varphi$
- contradiction, as $\mathcal{F}$ is not asymmetric
- hence φ does not exist

In general:

Corollary

Let C *be a class of frames, and let* $\mathcal{F}, \mathcal{F}'$ *be frames. If* $\mathcal{F} \in \mathsf{C}$, $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ *and* $\mathcal{F}' \notin \mathsf{C}$, *then* C *cannot be characterized by a modal formula.*

$\leftrightsquigarrow \subseteq \underline{\leftrightarrow}$?

What about the other direction: does modal equivalence of states imply that they are bisimilar?