

Meaning of Atoms

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How do you call them?

Models for propositional logic are called *valuations*.

Examples

Example

Some valuation Let $A = \{p, q, r\}$. Then a valuation v_1 might assign p to T , q to F and r to T .

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More formally

$p^{v_1} = T, q^{v_1} = F, r^{v_1} = T.$

write $v_1(p)$ instead of p^{v_1}

Building Propositions

We would like to build larger propositions, such as arguments, out of smaller ones, such as propositional atoms. We do this using *operators* that can be applied to propositions, and yield propositions.

Unary Operators

Let p be an atom.

All possibilities

The following options exist:

① $p^v = F: (op(p))^v = F.$

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The fourth operator *negates* its argument, T becomes F and F becomes T . We call this operator *negation*, and write $\neg p$ (pronounced “not p”).

Nullary Operators are Constants

The constant \top

The constant \top always evaluates to T , regardless of the valuation.

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The constant \top

The constant \top always evaluates to T , regardless of the valuation.

The constant \perp

The constant \perp always evaluates to F , regardless of the valuation.

Binary Operators: 16 choices

p	q	$op_1(p, q)$	$op_2(p, q)$	$op_3(p, q)$	$op_4(p, q)$
F	F	F	F	F	F
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_5(p, q)$	$op_6(p, q)$	$op_7(p, q)$	$op_8(p, q)$
F	F	F	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_9(p, q)$	$op_{10}(p, q)$	$op_{11}(p, q)$	$op_{12}(p, q)$
F	F	T	T	T	T
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_{13}(p, q)$	$op_{14}(p, q)$	$op_{15}(p, q)$	$op_{16}(p, q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Three Famous Ones

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op_{14} : $op_{14}(p, q)$ is T when p is F or q is T , and F otherwise.

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op_{14} : $op_{14}(p, q)$ is T when p is F or q is T , and F otherwise. Called *implication*, denoted $p \rightarrow q$.

Inductive Definition

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the least set F that fulfills the following rules:

- The constant symbols \perp and \top are in F .
- Every element of A is in F .
- If ϕ is in F , then $(\neg\phi)$ is also in F .
- If ϕ and ψ are in F , then $(\phi \wedge \psi)$ is also in F .
- If ϕ and ψ are in F , then $(\phi \vee \psi)$ is also in F .
- If ϕ and ψ are in F , then $(\phi \rightarrow \psi)$ is also in F .

Example

$$(((\neg p) \wedge q) \rightarrow (T \wedge (q \vee (\neg r))))$$

is a well-formed formula in propositional logic.

More Compact in BNF

$$\phi ::= p \mid \perp \mid \top \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

(Backus Naur Form)

Convention

The negation symbol \neg binds more tightly than \wedge and \vee , and \wedge and \vee bind more tightly than \rightarrow . Moreover, \rightarrow is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

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Example

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

can be written as

$$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$$

- 1 Atoms and Propositions
- 2 Semantics of Propositional Logic
 - Operations on Truth Values
 - Evaluation of Formulas
- 3 Proof Theory
- 4 Soundness and Completeness (preview)

Negating Truth Values

Definition

Function $\neg : \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B	$\neg B$
F	T
T	F

Conjunction of Truth Values

Definition

Function $\& : \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \& B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction of Truth Values

Definition

Function $|$: $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Implication of Truth Values

Definition

Function $\Rightarrow: \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \Rightarrow B_2$
F	F	T
F	T	T
T	F	F
T	T	T

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v , denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant \perp , then $v(\phi) = F$.
- If ϕ is the constant \top , then $v(\phi) = T$.
- If ϕ is an propositional atom p , then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \neg v(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \vee \tau)$, then $v(\phi) = v(\psi) | v(\tau)$.
- If ϕ has the form $(\psi \rightarrow \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid Formulas

Definition

A formula is called *valid* if it evaluates to T with respect to every possible valuation.

Examples

Example

Is

$$(((\neg p) \wedge q) \rightarrow (T \wedge (q \vee (\neg r))))$$

valid?

Examples

Example

Is

$$(((\neg p) \wedge q) \rightarrow (\top \wedge (q \vee (\neg r))))$$

valid?

Example

Find a valid formula that contains the propositional atoms p , q , r and w .

- 1 Atoms and Propositions
- 2 Semantics of Propositional Logic
- 3 Proof Theory**
 - Sequents
 - Axioms
 - Derived Rules
- 4 Soundness and Completeness (preview)

Sequents

Definition

A sequent consists of propositional formulas $\phi_1, \phi_2, \dots, \phi_n$, called *premises*, where $n \geq 0$, and a propositional formula ψ called *conclusion*. We write a sequent as follows:

$$\phi_1, \phi_2, \dots, \phi_n \vdash \psi$$

and say “ ψ is provable using the premises $\phi_1, \phi_2, \dots, \phi_n$ ”.

Introducing \top

$$\frac{}{\top} [\top i]$$

Rules for Conjunction

$$\frac{\phi \quad \psi}{\phi \wedge \psi} [\wedge i]$$

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$$\frac{\phi \quad \psi}{\phi \wedge \psi} [\wedge i]$$

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$$\frac{\phi \wedge \psi}{\psi} [\wedge e_2]$$

Example

$$p \wedge q, r \vdash q \wedge r$$

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Proof (graphical notation):

$$\frac{\frac{p \wedge q}{q} [\wedge e_2] \quad r}{q \wedge r} [\wedge i]$$

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Proof (text-based notation):

1	$(p \wedge q)$	premise
2	q	$\wedge e$ 1
3	r	premise
4	$q \wedge r$	$\wedge i$ 2,3

Double Negation Elimination

$$\frac{\neg\neg\phi}{\phi} [\neg\neg I]$$

Implication Elimination

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} [\rightarrow e]$$

We would like...

...to be able to prove:

$$p \rightarrow q \vdash \neg\neg p \rightarrow \neg q$$

A proof should look like this

$$p \rightarrow q \vdash \neg\neg p \rightarrow q$$

1	$p \rightarrow q$	premise
2	$\neg\neg p$	assumption
3	p	$\neg\neg e$ 2
4	q	$\rightarrow e$ 1, 3
5	$\neg\neg p \rightarrow q$	\rightarrow_i 2-4

Implication Introduction

$$\frac{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} [\rightarrow i]$$

Rules for Disjunction

$$\frac{\phi}{\phi \vee \psi} [\vee i_1]$$

$$\frac{\psi}{\phi \vee \psi} [\vee i_2]$$

Rules for Disjunction

$$\frac{\phi}{\phi \vee \psi} [\vee i_1] \qquad \frac{\psi}{\phi \vee \psi} [\vee i_2]$$

$$\frac{\phi \vee \psi \quad \begin{array}{|c|} \hline \phi \\ \vdots \\ \chi \\ \hline \end{array} \quad \begin{array}{|c|} \hline \psi \\ \vdots \\ \chi \\ \hline \end{array}}{\chi} [\vee e]$$

Axioms for \perp and Negation

$$\frac{\perp}{\phi} [\perp e]$$

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$$\frac{\perp}{\phi} [\perp e] \qquad \frac{\phi \quad \neg\phi}{\perp} [\neg e]$$

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$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \perp \end{array}}}{\neg\phi} [\neg i]$$

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ :

$$\phi \vdash \neg\neg\phi$$

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Lemma ($\neg\neg i$)

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Proof:

1	ϕ	premise
2	$\neg\phi$	assumption
3	\perp	$\neg e$ 1,2
4	$\neg\neg\phi$	$\neg i$ 2–3

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ :

$$\phi \vdash \neg\neg\phi$$

can be written like an axiom:

$$\frac{\phi}{\neg\neg\phi} [\neg\neg i]$$

Law of Excluded Middle

Lemma (LEM)

$$\frac{}{\phi \vee \neg \phi} [LEM]$$

- 1 Atoms and Propositions
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- 4 Soundness and Completeness (preview)
 - Entailment
 - Soundness and Completeness

Entailment

Definition

If, for all valuations in which all $\phi_1, \phi_2, \dots, \phi_n$ evaluate to \top , the formula ψ evaluates to \top as well, we say that $\phi_1, \phi_2, \dots, \phi_n$ semantically entail ψ , written:

$$\phi_1, \phi_2, \dots, \phi_n \models \psi$$

Soundness and Completeness

Theorem (Soundness of Propositional Logic)

Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional formulas.

If $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \dots, \phi_n \models \psi$.

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Theorem (Completeness of Propositional Logic)

Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional formulas.

If $\phi_1, \phi_2, \dots, \phi_n \models \psi$, then $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$.