Atoms

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More formally, we fix a set A of propositional atoms.

Meaning of Atoms

Models assign truth values

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More formally

A model (valuation) for a propositional logic for the set A of atoms is a mapping from A to $\{T, F\}$.

Inductive Definition

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the least set F that fulfills the following rules:

- The constant symbols $\bot$ and $\top$ are in F.
- Every element of A is in F.
- If ϕ is in F, then $(\neg\phi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \wedge \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \vee \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \rightarrow \psi)$ is also in F.

Parse trees

A formula

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r))))$$

Parse trees

A formula

$$(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$$

...and its parse tree:

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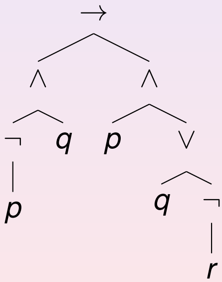

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\bot$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \vee \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \rightarrow \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid and Satisfiable Formulas

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A formula is called *satisfiable* if it evaluates to T with respect to at least one valuation.

Questions about Propositional Formula

- Is a given formula valid?
- Is a given formula satisfiable?
- Is a given formula invalid?
- Is a given formula unsatisfiable?
- Are two formulas equivalent?

Decision Problems

Definition

A *decision problem* is a question in some formal system with a yes-or-no answer.

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Examples

The question whether a given propositional formula is satisifiable (unsatisfiable, valid, invalid) is a decision problem.

The question whether two given propositional formulas are equivalent is also a decision problem.

How to Solve the Decision Problem?

Question

How do you decide whether a given propositional formula is satisfiable/valid?

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How do you decide whether a given propositional formula is satisfiable/valid?

The good news

We can construct a truth table for the formula and check if some/all rows have $\mathtt{T}$ in the last column.

Satisifiability is Decidable

An algorithm for satisifiability

Using a truth table, we can implement an *algorithm* that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

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Decision problems for which there is an algorithm computing "yes" whenever the answer is "yes", and "no" whenever the answer is "no", are called *decidable*.

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Decidability of satisfiability

The question, whether a given propositional formula is satisifiable, is decidable.

Is there a *practical* way of deciding satisfiability?

Question

Is there an *efficient* algorithm that decides whether a given formula is satisfiable?

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Question

Is there an *efficient* algorithm that decides whether a given formula is satisfiable?

More precisely...

Is there a *polynomial-time* algorithm that decides whether a given formula is satisfiable?

Is there a *practical* way of deciding satisfiability?

Question

Is there an *efficient* algorithm that decides whether a given formula is satisfiable?

More precisely...

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Answer

We do not know!

What *do* we know about satisfiability?

Truth assignment as witness

If the answer is “yes”, then a satisfying truth assignment can serve as a proof that the answer is indeed “yes”.

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Such a proof is called a *witness*.

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Witness for satisfiability

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Checking the witness

We can quickly check whether indeed the witness assignment makes the formula true. This can be done in time proportional to the size of the formula.

Conjunctive Normal Form

Definition

A literal L is either an atom p or the negation of an atom $\neg p$.
A formula C is in *conjunctive normal form* (CNF) if it is a conjunction of clauses, where each clause is a disjunction of literals:

$$\begin{aligned} L &::= p \mid \neg p \\ D &::= L \mid L \vee D \\ C &::= D \mid D \wedge C \end{aligned}$$

Examples

$(\neg p \vee q \vee r) \wedge (\neg q \vee r) \wedge (\neg r)$ is in CNF.
$(\neg p \vee q \vee r) \wedge ((p \wedge \neg q) \vee r) \wedge (\neg r)$ is not in CNF.
$(\neg p \vee q \vee r) \wedge \neg(\neg q \vee r) \wedge (\neg r)$ is not in CNF.

Usefulness of CNF

Lemma

A disjunction of literals $L_1 \vee L_2 \vee \cdots \vee L_m$ is valid iff there are $1 \leq i, j \leq m$ such that L_i is $\neg L_j$.

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Use lemma to prove all of:

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Proposition

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Satisfiability test

We can test satisfiability of ϕ by transforming $\neg\phi$ into CNF, and show that some clause is not valid.

Transformation to CNF

Theorem

Every formula in the propositional calculus can be transformed into an equivalent formula in CNF.

Algorithm for CNF Transformation

1. Eliminate implication using:
   $A \rightarrow B \equiv \neg A \vee B$
2. Push all negations inward using De Morgan's laws:
   $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$
   $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$
3. Eliminate double negations using the equivalence $\neg\neg A \equiv A$
4. The formula now consists of disjunctions and conjunctions of literals. Use the distributive laws
   $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
   $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$
   to eliminate conjunctions within disjunctions.

Example

$$
\begin{aligned}
(\neg p \rightarrow \neg q) \rightarrow (p \rightarrow q) &\equiv \neg(\neg\neg p \vee \neg q) \vee (\neg p \vee q) \\
&\equiv (\neg\neg\neg p \wedge q) \vee (\neg p \vee q) \\
&\equiv (\neg p \wedge q) \vee (\neg p \vee q) \\
&\equiv (\neg p \vee \neg p \vee q) \wedge (q \vee \neg p \vee q)
\end{aligned}
$$

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- Transform ψ into DAG; return “yes” for some satisfiable formulas, return “no” for some unsatisfiable formulas, return “don’t know” otherwise; example: propagation-based linear solver