Slides thanks to

- Martin Henz Aquinas Hobor
- CS 3234: Logic and Formal Systems

Necessity

- You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
  - Maybe the cook did it before dinner?
  - Maybe the maid did it after dinner?
- But: "The victim Ms Smith made a phone call *before* she was killed." is *necessarily* true.
- "Necessarily" means in all possible scenarios (worlds) under consideration.

Notions of Truth

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- We need to consider *modalities* if truth, such as:
  - necessity (“in all possible scenarios”)
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  - time (“forever in the future”)
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

1 Motivation

2 Basic Modal Logic
- Syntax
- Semantics
- Equivalences

3 Logic Engineering

Syntax of Basic Modal Logic

$$
\begin{aligned}
\phi \quad ::= \quad & \top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \wedge \phi) \\
& \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\
& \mid (\Box\phi) \mid (\Diamond\phi)
\end{aligned}
$$

Pronunciation and Examples

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Examples

$$(p \land \Diamond(p \to \Box\neg r))$$

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Examples

$$(p \wedge \Diamond(p \rightarrow \Box\neg r))$$

$$\Box((\Diamond q \wedge \neg r) \rightarrow \Box p)$$

Kripke Models

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2. a relation R on W, meaning $R \subseteq W \times W$, called the *accessibility relation*;
3. a function $L : W \to A \to \{T, F\}$, called *labeling function*.

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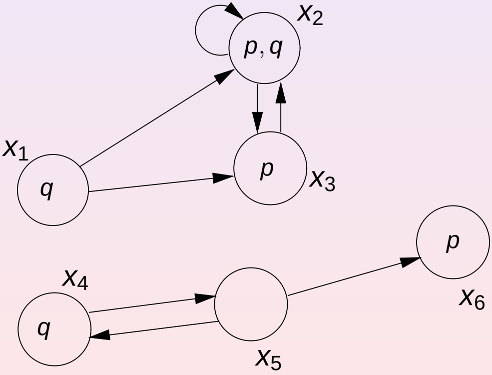
Contributions include modal logic, naming, belief, truth, the meaning of "I"

Example

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$
$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\}$$
$$L = \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}$$



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- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
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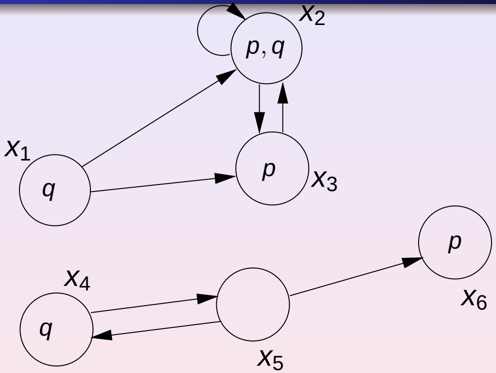
- $x \Vdash \Box\phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$
- $x \Vdash \Diamond\phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \phi$.

Example



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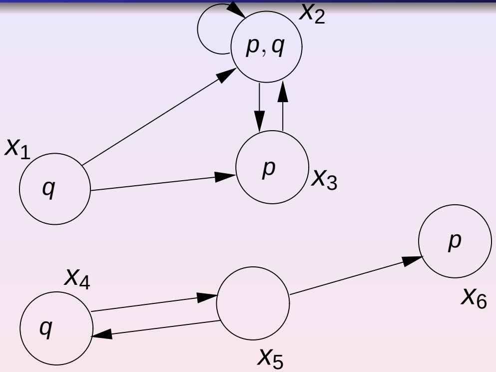
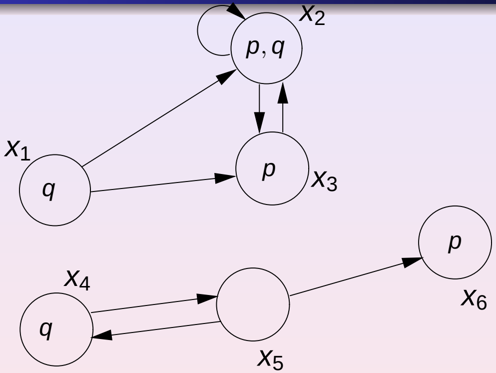


- $x_1 \Vdash q$

Example



- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$, $x_1 \nVdash \Box q$

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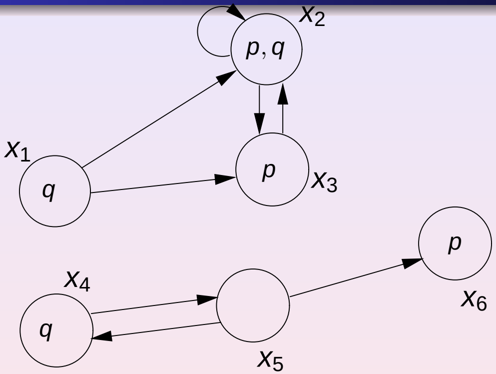

- $x_1 \Vdash q$
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- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$

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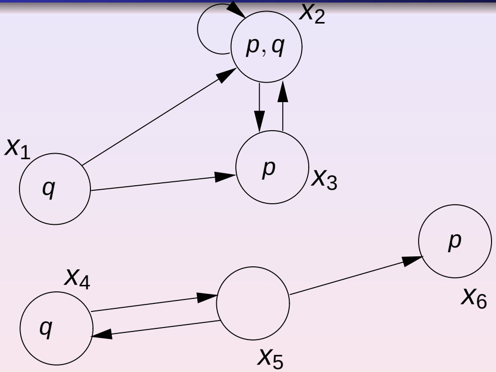

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- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$
- $x_6 \Vdash \Box\phi$ holds for all ϕ, but $x_6 \nVdash \Diamond\phi$ regardless of ϕ

Formula Schemes

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Greek letters denote formulas, and are not propositional atoms.

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Notation

Greek letters denote formulas, and are not propositional atoms.

Formula schemes

Terms where Greek letters appear instead of propositional atoms are called *formula schemes*.

Entailment and Equivalence

Definition

A set of formulas Γ entails a formula ψ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, whe have $x \Vdash \psi$ whenever $x \Vdash \phi$ for all $\phi \in \Gamma$. We say Γ entails ψ and write $\Gamma \models \psi$.

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Equivalence

We write $\phi \equiv \psi$ if $\phi \models \psi$ and $\psi \models \phi$.

Some Equivalences

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- Distributivity of $\Diamond$ over $\vee$:

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- $\Box\top \equiv \top$, $\Diamond\bot \equiv \bot$

Validity

Definition

A formula ϕ is valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

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- $\Box(\phi \land \psi) \to \Box\phi \land \Box\psi$
- $\Diamond(\phi \lor \psi) \to \Diamond\phi \lor \Diamond\psi$
- Formula K: $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$.

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A Range of Modalities

From the meaning of $\Box\phi$, we can conclude the meaning of $\Diamond\phi$, since $\Diamond\phi \equiv \neg\Box\neg\phi$:

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Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ
After any run of P, ϕ holds.	After some run of P, ϕ holds

Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \to \phi$	$\Box\phi \to \Box\Box\phi$	$\Diamond\phi \to \Box\Diamond\phi$	$\Diamond\top$	$\Box\phi \to \Diamond\phi$	$\Box\phi \vee \Box\neg\phi$	$\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi$	$\Diamond\phi \wedge \Diamond\psi \to \Diamond(\phi \wedge \psi)$
It is necessary that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
It will always be that ϕ	$\times$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$
It ought to be that ϕ	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q believes that ϕ	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q knows that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
After running P, ϕ	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$

Modalities lead to Interpretations of R

$\Box\phi$	$R(x,y)$
It is necessarily true that ϕ	y is possible world according to info at x
It will always be true that ϕ	y is a future world of x
It ought to be that ϕ	y is an acceptable world according to the information at x
Agent Q believes that ϕ	y could be the actual world according to Q's beliefs at x
Agent Q knows that ϕ	y could be the actual world according to Q's knowledge at x
After any execution of P, ϕ holds	y is a possible resulting state after execution of P at x

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- serial: for every x there is a y such that $R(x, y)$.
- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.

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- Euclidean: for every $x, y, z \in W$ with $R(x,y)$ and $R(x,z)$, we have $R(y,z)$.

Possible Properties of R

- reflexive: for every $w \in W$, we have $R(x, x)$.
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- equivalence: reflexive, symmetric and transitive.

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- Should R be serial?

Necessarily true and Reflexivity

Guess

R is reflexive if and only if $\Box\phi \to \phi$ is valid.

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- Ignore L, and only consider the (W, R) part of a model, called *frame*.
- Establish formula schemes based on properties of frames.

Reflexivity and Transitivity

Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

- R is reflexive;
- $\mathcal{F}$ satisfies $\Box\phi \to \phi$;
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Theorem 2

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$x \Vdash \Box\phi \to \phi$ Suppose $x \Vdash \Box\phi$.
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Using the semantics of $\to$: $x \Vdash \Box\phi \to \phi$

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$2 \Rightarrow 3$: Just set ϕ to be p

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Proof by contradiction: Assume $(x, x) \notin R$. Then we would have $x \Vdash \Box p$, but not $x \Vdash p$.
Contradiction!

Formula Schemes and Properties of R

name	formula scheme	property of R
T	$\Box\phi \to \phi$	reflexive
B	$\phi \to \Box\Diamond\phi$	symmetric
D	$\Box\phi \to \Diamond\phi$	serial
4	$\Box\phi \to \Box\Box\phi$	transitive
5	$\Diamond\phi \to \Box\Diamond\phi$	Euclidean
	$\Box\phi \to \Diamond\phi \wedge \Diamond\phi \to \Box\phi$	functional
	$\Box(\phi \wedge \Box\phi \to \psi) \vee \Box(\psi \wedge \Box\psi \to \phi)$	linear

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- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let $\mathcal{L}_c$ be the smallest closed superset of $\mathcal{L}$.
- Γ entails ψ in $\mathcal{L}$ iff $\Gamma \cup \mathcal{L}_c$ semantically entails ψ. We say $\Gamma \models_{\mathcal{L}} \psi$.

Examples of Modal Logics: K

K is the weakest modal logic, $\mathcal{L} = \emptyset$.

Examples of Modal Logics: KT45

$$\mathcal{L} = \{T, 4, 5\}$$

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Used for reasoning about knowledge.

- T: Truth: agent Q only knows true things.
- 4: Positive introspection: If Q knows something, he knows that he knows it.
- 5: Negative introspection: If Q doesn't know something, he knows that he doesn't know it.