

# Slides thanks to

- Martin Henz Aquinas Hobor
- CS 3234: Logic and Formal Systems

# Predicates

## Example

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- $S(\text{andy})$  could denote that Andy is a student.
- $I(\text{paul})$  could denote that Paul is an instructor.
- $Y(\text{andy}, \text{paul})$  could denote that Andy is younger than Paul.

# Example

English

Every girl is younger than her mother.

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## Predicates

$G(x)$ :  $x$  is a girl

$M(x, y)$ :  $x$  is  $y$ 's mother

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## The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(y, x) \rightarrow Y(x, y))$$

# A “Mother” Function

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The sentence using a function

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$



# Predicate Vocabulary

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- a set of predicate symbols  $\mathcal{P}$
- a set of function symbols  $\mathcal{F}$

# Arity of Functions and Predicates

Every function symbol in  $\mathcal{F}$  and predicate symbol in  $\mathcal{P}$  comes with a fixed arity, denoting the number of arguments the symbol can take.

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# Terms

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- $f$  ranges over function symbols in  $\mathcal{F}$  with arity  $n > 0$ .

# Examples of Terms

If  $n$  is nullary,  $f$  is unary, and  $g$  is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

# Formulas

$$\phi ::= P(t, \dots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ (\phi \rightarrow \phi) \mid (\forall x\phi) \mid (\exists x\phi)$$

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- $x$  are variables in  $\mathcal{V}$ .

# Equality as Predicate

Equality is a common predicate, usually used in infix notation.

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## Example

Instead of the formula

$$= (f(x), g(x))$$

we usually write the formula

$$f(x) = g(x)$$

# Models

## Definition

Let  $\mathcal{F}$  contain function symbols and  $\mathcal{P}$  contain predicate symbols. A model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  consists of:

- 1 A non-empty set  $A$ , the *universe*;
- 2 for each nullary function symbol  $f \in \mathcal{F}$  a concrete element  $f^{\mathcal{M}} \in A$ ;
- 3 for each  $f \in \mathcal{F}$  with arity  $n > 0$ , a concrete function  $f^{\mathcal{M}} : A^n \rightarrow A$ ;
- 4 for each  $P \in \mathcal{P}$  with arity  $n > 0$ , a function  $P^{\mathcal{M}} : U^n \rightarrow \{F, T\}$ .
- 5 for each  $P \in \mathcal{P}$  with arity  $n = 0$ , a value from  $\{F, T\}$ .

# Equality Revisited

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## Extensionality restriction

This means that allowable models are restricted to those in which  $a =^{\mathcal{M}} b$  holds if and only if  $a$  and  $b$  are the same elements of the model's universe.

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- in case  $\phi$  is of the form  $P$ , if  $P^{\mathcal{M}} = T$ ;
- in case  $\phi$  has the form  $\forall x\psi$ , if the  $\mathcal{M} \models_{l[x \mapsto a]} \psi$  holds for all  $a \in A$ ;



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- in case  $\phi$  has the form  $\exists x\psi$ , if the  $\mathcal{M} \models_{l[x \mapsto a]} \psi$  holds for some  $a \in A$ ;

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- in case  $\phi$  has the form  $\psi_1 \rightarrow \psi_2$ , if  $\mathcal{M} \models_I \psi_1$  holds whenever  $\mathcal{M} \models_I \psi_2$  holds.

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$\Gamma \models \psi$  iff for all models  $\mathcal{M}$  and environments  $I$ , whenever  $\mathcal{M} \models_I \phi$  holds for all  $\phi \in \Gamma$ , then  $\mathcal{M} \models_I \psi$ .

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$\psi$  is satisfiable iff there is some model  $\mathcal{M}$  and some environment  $I$  such that  $\mathcal{M} \models_I \psi$  holds.



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## Satisfiability of Formula Sets

$\Gamma$  is satisfiable iff there is some model  $\mathcal{M}$  and some environment  $I$  such that  $\mathcal{M} \models_I \phi$ , for all  $\phi \in \Gamma$ .

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Let  $\Gamma$  be a possibly infinite set of formulas in predicate logic and  $\psi$  a formula.

## Validity

$\psi$  is valid iff for all models  $\mathcal{M}$  and environments  $I$ , we have  $\mathcal{M} \models_I \psi$ .

# The Problem with Predicate Logic

## Entailment ranges over models

Semantic entailment between sentences:  $\phi_1, \phi_2, \dots, \phi_n \models \psi$   
requires that in *all* models that satisfy  $\phi_1, \phi_2, \dots, \phi_n$ , the  
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Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

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## Idea from propositional logic

Can we use natural deduction for showing entailment?

- 1 Review: Syntax and Semantics
- 2 Proof Theory**
  - Equality
  - Universal Quantification
  - Existential Quantification
- 3 Equivalences and Properties

# Natural Deduction for Predicate Logic

## Relationship between propositional and predicate logic

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## Example

$$\frac{\phi \quad \psi}{\phi \wedge \psi} [\wedge i]$$

## Built-in Rules for Equality

$$\frac{}{t = t} [= i] \qquad \frac{t_1 = t_2 \quad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi} [= e]$$

# Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

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- |   |                     |           |
|---|---------------------|-----------|
| 1 | $f(x) = g(x)$       | premise   |
| 2 | $h(f(x)) = h(f(x))$ | $= i$     |
| 3 | $h(g(x)) = h(f(x))$ | $= e$ 1,2 |

# Elimination of Universal Quantification

$$\frac{\forall x \phi}{[x \Rightarrow t] \phi} [\forall x e]$$

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# Example

$$\frac{\forall x \phi}{[x \Rightarrow t] \phi} [\forall x e]$$

We prove:  $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$



# Example

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|---|--|---------------------|
| 1 | $S(g(john))$                             | premise             |
| 2 | $\forall x(S(x) \rightarrow \neg L(x))$  | premise             |
| 3 | $S(g(john)) \rightarrow \neg L(g(john))$ | $\forall x e$ 2     |
| 4 | $\neg L(g(john))$                        | $\rightarrow e$ 3,1 |

# Introduction of Universal Quantification

$$\frac{\begin{array}{c} \boxed{\begin{array}{c} \vdots \\ [x \Rightarrow x_0] \phi \end{array}}^{x_0} \\ \hline [\forall x i] \\ \forall x \phi \end{array}}{\forall x \phi}$$

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If we manage to establish a formula  $\phi$  about a fresh variable  $x_0$ , we can assume  $\forall x \phi$ .

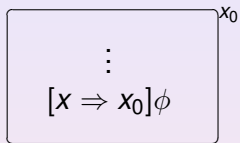
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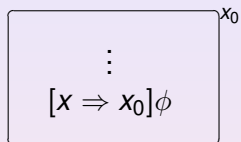
The variable  $x_0$  must be *fresh*; we cannot introduce the same variable twice in nested boxes.

# Example



$$\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x) \text{ via } \frac{\quad}{\forall x\phi}$$

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$\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x)$  via  $\frac{\quad}{\forall x\phi}$

1	$\forall x(P(x) \rightarrow Q(x))$	premise
2	$\forall xP(x)$	premise
3	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 1$ <span style="float: right;"><math>x_0</math></span>
4	$P(x_0)$	$\forall x e 2$
5	$Q(x_0)$	$\rightarrow e 3,4$
6	$\forall xQ(x)$	$\forall x i 3-5$

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In order to prove  $\exists x \phi$ , it suffices to find a term  $t$  as “witness”, provided that  $t$  is free for  $x$  in  $\phi$ .

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- 1 A *non-empty* set  $U$ , the *universe*;
- 2 ...

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## Remark

Compare this with Traditional Logic (Coq Quiz 1).

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Because  $U$  must not be empty, we should be able to prove the sequent above.

## Example (continued)

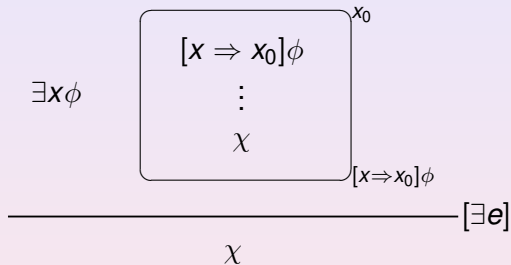
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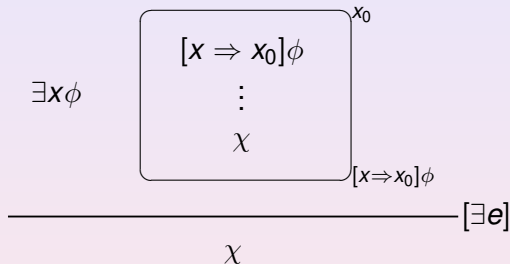
1	$\forall x\phi$	premise
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3	$\exists x\phi$	$\exists x i 2$

# Elimination of Existential Quantification





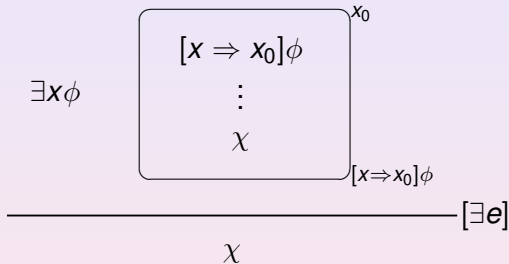
# Elimination of Existential Quantification



## Making use of $\exists$

If we know  $\exists x \phi$ , we know that there exist at least one object  $x$  for which  $\phi$  holds.

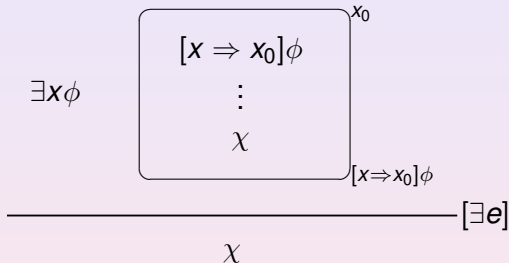
# Elimination of Existential Quantification



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If we know  $\exists x \phi$ , we know that there exist at least one object  $x$  for which  $\phi$  holds. We call that element  $x_0$ , and assume  $[x \Rightarrow x_0] \phi$ .

# Elimination of Existential Quantification



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If we know  $\exists x \phi$ , we know that there exist at least one object  $x$  for which  $\phi$  holds. We call that element  $x_0$ , and assume  $[x \Rightarrow x_0] \phi$ . Without assumptions on  $x_0$ , we prove  $\chi$  ( $x_0$  not in  $\chi$ ).

## Example

$$\forall x(P(x) \rightarrow Q(x)), \exists xP(x) \vdash \exists xQ(x)$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\exists xP(x)$	premise	
3	$P(x_0)$	assumption	$x_0$
4	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 1$	
5	$Q(x_0)$	$\rightarrow e 4,3$	
6	$\exists xQ(x)$	$\exists x i 5$	
7	$\exists xQ(x)$	$\exists x e 2,3-6$	

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6	$\exists xQ(x)$	$\exists x i 5$	
7	$\exists xQ(x)$	$\exists x e 2,3-6$	

Note that  $\exists xQ(x)$  within the box does not contain  $x_0$ , and therefore can be “exported” from the box.

## Another Example

1	$\forall x(Q(x) \rightarrow R(x))$	premise	
2	$\exists x(P(x) \wedge Q(x))$	premise	
3	$P(x_0) \wedge Q(x_0)$	assumption	$x_0$
4	$Q(x_0) \rightarrow R(x_0)$	$\forall x e 1$	
5	$Q(x_0)$	$\wedge e_2 3$	
6	$R(x_0)$	$\rightarrow e 4,5$	
7	$P(x_0)$	$\wedge e_1 3$	
8	$P(x_0) \wedge R(x_0)$	$\wedge i 7, 6$	
9	$\exists x(P(x) \wedge R(x))$	$\exists x i 8$	
10	$\exists x(P(x) \wedge R(x))$	$\exists x e 2,3-9$	

## Variables must be fresh! This is not a proof!

- 1  $\exists xP(x)$  premise  
2  $\forall x(P(x) \rightarrow Q(x))$  premise

3			$x_0$
4	$P(x_0)$	assumption	$x_0$
5	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 2$	
6	$Q(x_0)$	$\rightarrow e 5,4$	
7	$Q(x_0)$	$\exists x e 1, 4-6$	
8	$\forall yQ(y)$	$\forall y i 3-7$	

- 1 Review: Syntax and Semantics
- 2 Proof Theory
- 3 Equivalences and Properties**
  - Quantifier Equivalences
  - Soundness and Completeness
  - Undecidability, Compactness



# Equivalences

## Two-way-provable

We write  $\phi \dashv\vdash \psi$  iff  $\phi \vdash \psi$  and also  $\psi \vdash \phi$ .

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$$\begin{aligned}\neg\forall x\phi &\dashv\vdash \exists x\neg\phi \\ \neg\exists x\phi &\dashv\vdash \forall x\neg\phi \\ \forall x\forall y\phi &\dashv\vdash \forall y\forall x\phi\end{aligned}$$

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$$\exists x \exists y \phi \dashv\vdash \exists y \exists x \phi$$

$$\forall x \phi \wedge \forall x \psi \dashv\vdash \forall x (\phi \wedge \psi)$$

$$\exists x \phi \vee \exists x \psi \dashv\vdash \exists x (\phi \vee \psi)$$

$$\neg \forall x \phi \vdash \exists x \neg \phi$$

1	$\neg \forall x \phi$	premise
2	$\neg \exists x \neg \phi$	assumption
3		$x_0$
4	$\neg [x \Rightarrow x_0] \phi$	assumption
5	$\exists x \neg \phi$	$\exists x \ i \ 4$
6	$\perp$	$\neg e \ 5, 2$
7	$[x \Rightarrow x_0] \phi$	PBC 4–6
8	$\forall x \phi$	$\forall x \ i \ 3\text{--}7$
9	$\perp$	$\neg e \ 8, 1$
10	$\exists x \neg \phi$	PBC 2–9



$$\exists x \exists y \phi \vdash \exists y \exists x \phi$$

Assume that  $x$  and  $y$  are different variables.

$\exists x \exists y \phi \vdash \exists y \exists x \phi$ 

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1	$\exists x \exists y \phi$	premise	
2	$[x \Rightarrow x_0](\exists y \phi)$	assumption	$x_0$
3	$\exists y([x \Rightarrow x_0]\phi$	def of subst ( $x, y$ different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	$y_0$
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst ( $x, y, x_0, y_0$ different)	
6	$\exists x[y \Rightarrow y_0]\phi$	$\exists x$ i 5	
7	$\exists y \exists x \phi$	$\exists y$ i 6	
8	$\exists y \exists x \phi$	$\exists y$ e 3, 4–7	
9	$\exists y \exists x \phi$	$\exists x$ e 1, 2–8	

# More Equivalences

Assume that  $x$  is not free in  $\psi$

$$\forall x\phi \wedge \psi \dashv\vdash \forall x(\phi \wedge \psi)$$

$$\forall x\phi \vee \psi \dashv\vdash \forall x(\phi \vee \psi)$$

$$\exists x\phi \wedge \psi \dashv\vdash \exists x(\phi \wedge \psi)$$

$$\exists x\phi \vee \psi \dashv\vdash \exists x(\phi \vee \psi)$$

# Central Result of Natural Deduction

$$\phi_1, \dots, \phi_n \models \psi$$

iff

$$\phi_1, \dots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation

# Recall: Decidability

## Decision problems

A *decision problem* is a question in some formal system with a yes-or-no answer.

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### Decidability of satisfiability

The question, whether a given propositional formula is satisfiable, is decidable.

# Undecidability of Predicate Logic

## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .



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- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .
- Establish that  $\models \phi$  holds if and only if  $C$  has a solution.
- Conclude that validity of predicate logic formulas is undecidable.

# Compactness

## Theorem

*Let  $\Gamma$  be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of  $\Gamma$  are satisfiable, the  $\Gamma$  itself is satisfiable.*

# Application of Compactness

## Theorem (Löwenheim-Skolem Theorem)

*Let  $\psi$  be a sentence of predicate logic such that for any natural number  $n \geq 1$  there is a model of  $\psi$  with at least  $n$  elements. Then  $\psi$  has a model with infinitely many elements.*

# Slides thanks to

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