Slides thanks to

- Martin Henz Aquinas Hobor
- CS 3234: Logic and Formal Systems

Predicates

Example

Every student is younger than *some* instructor.

Predicates

Example

Every student is younger than *some* instructor.

- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.

Example

English

Every girl is younger than her mother.

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Predicates

$G(x)$: x is a girl
$M(x, y)$: x is y's mother
$Y(x, y)$: x is younger than y

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$G(x)$: x is a girl

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The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(y, x) \rightarrow Y(x, y))$$

A “Mother” Function

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The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(y,x) \rightarrow Y(x,y))$$

The sentence using a function

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$

Predicate Vocabulary

At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols $\mathcal{P}$

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- a set of predicate symbols $\mathcal{P}$
- a set of function symbols $\mathcal{F}$

Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

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Special case: Nullary Predicates

Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.

Terms

$$t ::= \; x \mid c \mid f(t, \ldots, t)$$

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- c ranges over nullary function symbols in $\mathcal{F}$, and
- f ranges over function symbols in $\mathcal{F}$ with arity $n > 0$.

Examples of Terms

If n is nullary, f is unary, and g is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

Formulas

$$\begin{aligned}\phi \quad ::= \quad & P(t, \ldots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ & (\phi \rightarrow \phi) \mid (\forall x \phi) \mid (\exists x \phi)\end{aligned}$$

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where

- $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 0$,
- t are terms over $\mathcal{F}$ and $\mathcal{V}$, and
- x are variables in $\mathcal{V}$.

Equality as Predicate

Equality is a common predicate, usually used in infix notation.

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Example

Instead of the formula

$$=(f(x), g(x))$$

we usually write the formula

$$f(x) = g(x)$$

Models

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^{\mathcal{M}} : U^n \to \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$.

Equality Revisited

Interpretation of equality

Usually, we require that the equality predicate $=$ is interpreted as same-ness.

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Extensionality restriction

This means that allowable models are restricted to those in which $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe.

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- in case ϕ is of the form P, if $P^{\mathcal{M}} = T$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;

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- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;
- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for some $a \in A$;

Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_l \psi$ does not hold;

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Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_I \psi$ does not hold;
- in case ϕ has the form $\psi_1 \lor \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds or $\mathcal{M} \models_I \psi_2$ holds;
- in case ϕ has the form $\psi_1 \land \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds and $\mathcal{M} \models_I \psi_2$ holds; and

Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_l \psi$ does not hold;
- in case ϕ has the form $\psi_1 \vee \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds or $\mathcal{M} \models_l \psi_2$ holds;
- in case ϕ has the form $\psi_1 \wedge \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds and $\mathcal{M} \models_l \psi_2$ holds; and
- in case ϕ has the form $\psi_1 \rightarrow \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds whenever $\mathcal{M} \models_l \psi_2$ holds.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

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Entailment

$\Gamma \models \psi$ iff for all models $\mathcal{M}$ and environments l, whenever $\mathcal{M} \models_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \models_l \psi$.

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Satisfiability of Formula Sets

Γ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \phi$, for all $\phi \in \Gamma$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Validity

ψ is valid iff for all models $\mathcal{M}$ and environments l, we have $\mathcal{M} \models_l \psi$.

The Problem with Predicate Logic

Entailment ranges over models

Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ requires that in *all* models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence ψ is satisfied.

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How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

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Idea from propositional logic

Can we use natural deduction for showing entailment?

Natural Deduction for Predicate Logic

Relationship between propositional and predicate logic

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Example

$$\frac{\phi \qquad \psi}{\phi \land \psi}\,[\land i]$$

Built-in Rules for Equality

$$\frac{}{t = t}\,[= i] \qquad \frac{t_i = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}\,[= e]$$

Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

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1	$f(x) = g(x)$	premise
2	$h(f(x)) = h(f(x))$	$= i$
3	$h(g(x)) = h(f(x))$	$= e$ 1,2

Elimination of Universal Quantification

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}\,[\forall x\ e]$$

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Once you have proven $\forall x \phi$, you can replace x by any term t in ϕ, provided that t is free for x in ϕ.

Example

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}\,[\forall x\ e]$$

We prove: $S(g(john)), \forall x(S(x) \to \neg L(x)) \vdash \neg L(g(john))$

Example

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We prove: $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$

1	$S(g(john))$	premise
2	$\forall x(S(x) \rightarrow \neg L(x))$	premise
3	$S(g(john)) \rightarrow \neg L(g(john))$	$\forall x\ e$ 2
4	$\neg L(g(john))$	$\rightarrow e$ 3,1

Introduction of Universal Quantification

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If we manage to establish a formula ϕ about a fresh variable x_0, we can assume $\forall x \phi$.
The variable x_0 must be *fresh*; we cannot introduce the same variable twice in nested boxes.

Example

$$\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x) \text{ via } \frac{\boxed{\begin{array}{c} \vdots \\ [x \Rightarrow x_0]\phi \end{array}}^{\,x_0}}{\forall x \phi}$$

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1	$\forall x(P(x) \to Q(x))$	premise	
2	$\forall x P(x)$	premise	
3	$P(x_0) \to Q(x_0)$	$\forall x\ e$ 1	x_0
4	$P(x_0)$	$\forall x\ e$ 2	
5	$Q(x_0)$	$\to e$ 3,4	
6	$\forall x Q(x)$	$\forall x\ i$ 3–5	

Introduction of Existential Quantification

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Recall: Definition of Models

A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A *non-empty* set U, the *universe*;
2. ...

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Remark

Compare this with Traditional Logic (Coq Quiz 1).

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Because U must not be empty, we should be able to prove the sequent above.

Example (continued)

$$\forall x \phi \vdash \exists x \phi$$

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1	$\forall x \phi$	premise
2	$[x \Rightarrow x]\phi$	$\forall x\ e\ 1$
3	$\exists x \phi$	$\exists x\ i\ 2$

Elimination of Existential Quantification

$$\frac{\exists x \phi \qquad \boxed{\begin{array}{c}[x \Rightarrow x_0]\phi \\ \vdots \\ \chi\end{array}}^{x_0}_{[x\Rightarrow x_0]\phi}}{\chi}\,[\exists e]$$

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Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds.

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Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume $[x \Rightarrow x_0]\phi$.

Elimination of Existential Quantification

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\dfrac{\exists x \phi \qquad \boxed{\begin{array}{c} [x \Rightarrow x_0]\phi \\ \vdots \\ \chi \end{array}}^{\,x_0}_{\,[x \Rightarrow x_0]\phi}}{\chi}\ [\exists e]
$$

Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume $[x \Rightarrow x_0]\phi$. Without assumptions on x_0, we prove χ (x_0 not in χ).

Example

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\exists x P(x)$	premise	
3	$P(x_0)$	assumption	x_0
4	$P(x_0) \rightarrow Q(x_0)$	$\forall x$ *e* 1	
5	$Q(x_0)$	$\rightarrow$ *e* 4,3	
6	$\exists x Q(x)$	$\exists x$ *i* 5	
7	$\exists x Q(x)$	$\exists x$ *e* 2,3–6	

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6	$\exists x Q(x)$	$\exists x$ i 5	
7	$\exists x Q(x)$	$\exists x$ e 2,3–6	

Note that $\exists x Q(x)$ within the box does not contain x_0, and therefore can be “exported” from the box.

Another Example

1	$\forall x(Q(x) \rightarrow R(x))$	premise	
2	$\exists x(P(x) \wedge Q(x))$	premise	
3	$P(x_0) \wedge Q(x_0)$	assumption	x_0
4	$Q(x_0) \rightarrow R(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\wedge e_2$ 3	
6	$R(x_0)$	$\rightarrow e$ 4,5	
7	$P(x_0)$	$\wedge e_1$ 3	
8	$P(x_0) \wedge R(x_0)$	$\wedge i$ 7, 6	
9	$\exists x(P(x) \wedge R(x)$	$\exists x\ i$ 8	
10	$\exists x(P(x) \wedge R(x))$	$\exists x\ e$ 2,3–9	

Variables must be fresh! This is not a proof!

1	$\exists x P(x)$	premise	
2	$\forall x(P(x) \rightarrow Q(x))$	premise	
3			x_0
4	$P(x_0)$	assumption	x_0
5	$P(x_0) \rightarrow Q(x_0)$	$\forall x$ *e* 2	
6	$Q(x_0)$	$\rightarrow$ *e* 5,4	
7	$Q(x_0)$	$\exists x$ *e* 1, 4–6	
8	$\forall y Q(y)$	$\forall y$ *i* 3–7	

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

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Some simple equivalences

$$\neg\forall x\phi \quad \dashv\vdash \quad \exists x\neg\phi$$

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We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$\begin{aligned} \neg\forall x\phi &\dashv\vdash \exists x\neg\phi \\ \neg\exists x\phi &\dashv\vdash \forall x\neg\phi \end{aligned}$$

Equivalences

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We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$\begin{array}{rcl} \neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\ \neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\ \forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \end{array}$$

Equivalences

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\begin{aligned}
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\neg\exists x\phi &\dashv\vdash \forall x\neg\phi \\
\forall x\forall y\phi &\dashv\vdash \forall y\forall x\phi \\
\exists x\exists y\phi &\dashv\vdash \exists y\exists x\phi
\end{aligned}
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\neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\
\neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\
\forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\
\exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \\
\forall x\phi \wedge \forall x\psi & \dashv\vdash & \forall x(\phi \wedge \psi)
\end{array}
$$

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$
\begin{array}{rcl}
\neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\
\neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\
\forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\
\exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \\
\forall x\phi \wedge \forall x\psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\
\exists x\phi \vee \exists x\psi & \dashv\vdash & \exists x(\phi \vee \psi)
\end{array}
$$

$\neg\forall x\phi \vdash \exists x\neg\phi$

1	$\neg\forall x\phi$	premise
2	$\neg\exists x\neg\phi$	assumption
3		x_0
4	$\neg[x \Rightarrow x_0]\phi$	assumption
5	$\exists x\neg\phi$	$\exists x\ i$ 4
6	$\bot$	$\neg e$ 5, 2
7	$[x \Rightarrow x_0]\phi$	PBC 4–6
8	$\forall x\phi$	$\forall x\ i$ 3–7
9	$\bot$	$\neg e$ 8, 1
10	$\exists x\neg\phi$	PBC 2–9

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

Assume that x and y are different variables.

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

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1	$\exists x \exists y \phi$	premise	
2	$[x \Rightarrow x_0](\exists y \phi)$	assumption	x_0
3	$\exists y([x \Rightarrow x_0]\phi$	def of subst (x, y different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	y_0
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst (x, y, x_0, y_0 different)	
6	$\exists x[y \rightarrow y_0]\phi$	$\exists x$ *i* 5	
7	$\exists y \exists x \phi$	$\exists y$ *i* 6	
8	$\exists y \exists x \phi$	$\exists y$ *e* 3, 4–7	
9	$\exists y \exists x \phi$	$\exists x$ *e* 1, 2–8	

More Equivalences

Assume that x is not free in ψ

$$\begin{aligned}
\forall x \phi \wedge \psi &\dashv\vdash \forall x(\phi \wedge \psi) \\
\forall x \phi \vee \psi &\dashv\vdash \forall x(\phi \vee \psi) \\
\exists x \phi \wedge \psi &\dashv\vdash \exists x(\phi \wedge \psi) \\
\exists x \phi \vee \psi &\dashv\vdash \exists x(\phi \vee \psi)
\end{aligned}$$

Central Result of Natural Deduction

$$\phi_1, \ldots, \phi_n \models \psi$$

iff

$$\phi_1, \ldots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation

Recall: Decidability

Decision problems

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Decidability of satisfiability

The question, whether a given propositional formula is satisfiable, is decidable.

Undecidability of Predicate Logic

Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula ϕ in that language, decides whether $\models \phi$.

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Proof sketch

- Establish that the Post Correspondence Problem (PCP) is undecidable

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- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say C, to a formula ϕ.
- Establish that $\models \phi$ holds if and only if C has a solution.
- Conclude that validity of predicate logic formulas is undecidable.

Compactness

Theorem

Let Γ be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of Γ are satisfiable, the Γ itself is satisfiable.

Application of Compactness

Theorem (Löwenheim-Skolem Theorem)

Let ψ be a sentence of predicate logic such that for any natural number $n \geq 1$ there is a model of ψ with at least n elements. Then ψ has a model with infinitely many elements.

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