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# Spatial and Temporal Knowledge Representation

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PART III: Spatial Knowledge Representation

# Contents of Part III

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2. Top-down vs bottom-up theories of space
3. Mereotopology

# Analogies and Disanalogies between Space and Time

# How does time differ from space?

- **Dimension**
  Time has one dimension, space has three.
- **Direction**
  Time is ordered from past to future, space has no such ordering.
- **Transience**
  Time 'flows': successive moments pass from future to present to past; space is not like that.

# How is time similar to space?

- **Extension**
  Time and space are both extended.
- **Structure**
  Time has instants/intervals, space has points/regions.
- **Metricity**
  Extended portions of time and space can both be measured.
- **Occupancy**
  Time and space are both occupied: space by matter and objects, time by processes and events.

# Formal analogies

There are many **formal analogies** between space and time, e.g.,

| SPACE | TIME |
|---|---|
| instant | point |
| interval | region |
| endpoints of interval | boundary of region |
| event occurs in interval | object located in region |
| process goes on in interval | matter fills region |

**Common relations:** Overlap, adjacency, containment

**Common problems:** Discrete vs Continuous; Vagueness, Indeterminacy, Granularity

# Top-down vs bottom-up theories of space

# The Container Metaphor for Space

- We normally locate objects in relation to other objects, e.g.,

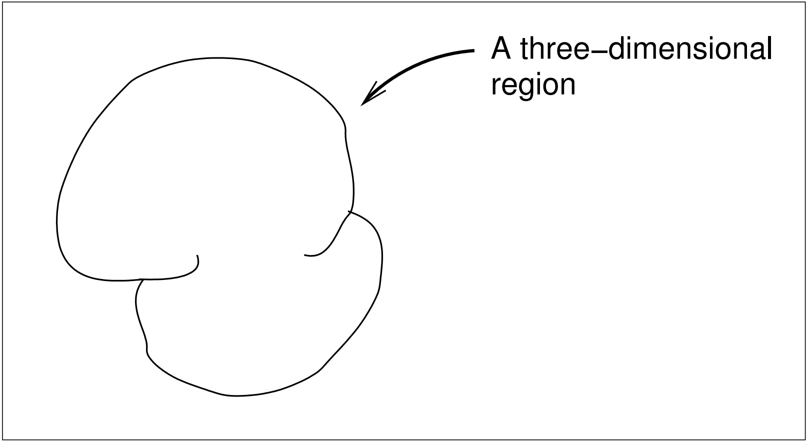
  *The key is in the box*
  *The box is on the table*
  *The table is in the bedroom*
  *The bedroom is above the living room*

- But we also say that objects are located **in space**. We can think of the position of an object as *that portion of space which it exactly fills* (at a given time).
- *The key is in the box*: the position of the key is a part of the region of space defined by the interior of the box.
- A widespread approach to formalising statements about location in space is to base it on a theory of spatial points and regions and their properties and interrelations.

# Top-down Theory of Space: 3D regions

- Start with three-dimensional spatial regions: these provide the locations for solid objects.

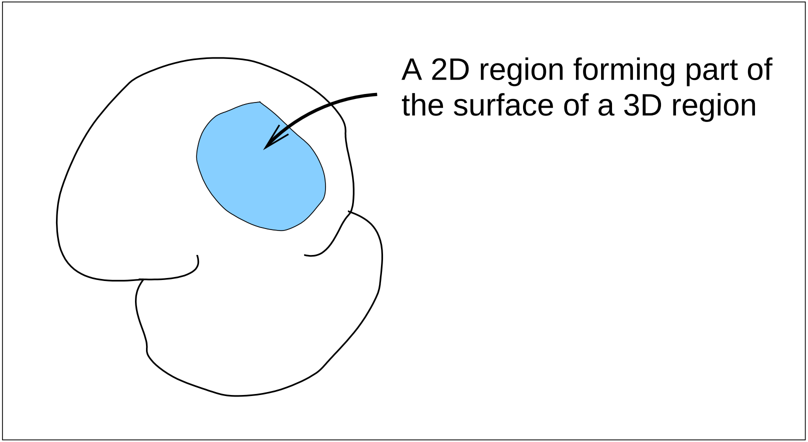
# Top-down Theory of Space: 3D regions



# Top-down Theory of Space: 2D regions

- Start with three-dimensional spatial regions: these provide the locations for solid objects.
- The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.

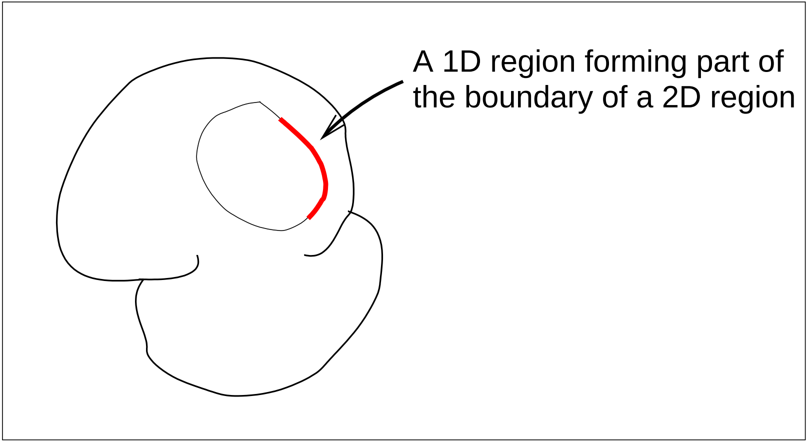
# Top-down Theory of Space: 2D regions



# Top-down Theory of Space: 1D regions

- Start with three-dimensional spatial regions: these provide the locations for solid objects.
- The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.
- The boundary of a 2D region is a one-dimensional region, which can be subdivided into other 1D regions.

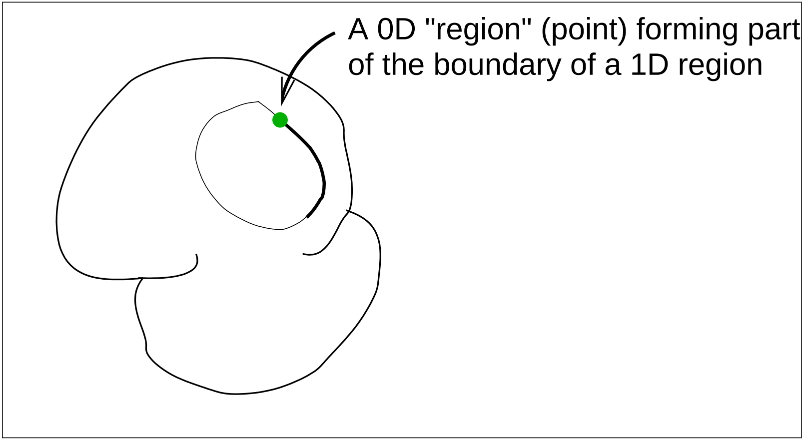
# Top-down Theory of Space: 1D regions



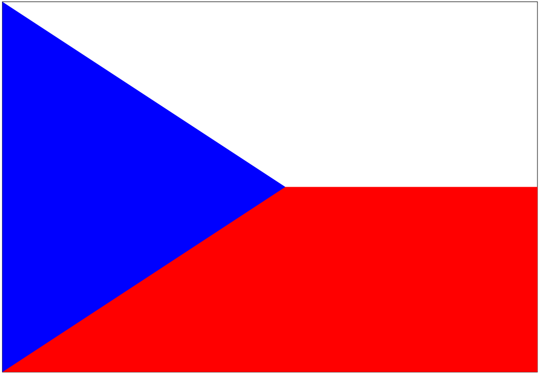
# Top-down Theory of Space: 0D "regions" (= points)

- Start with three-dimensional spatial regions: these provide the locations for solid objects.
- The surface of a 3D region is a two-dimensional region, which can be subdivided into other 2D regions.
- The boundary of a 2D region is a one-dimensional region, which can be subdivided into other 1D regions.
- The boundary of a 1D region is a set of points — zero-dimensional regions.

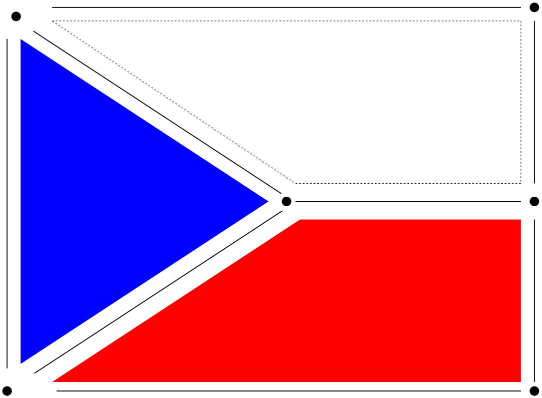
# Top-down Theory of Space: 0D regions



# Example: The Czech Flag



# Example: The Czech Flag dissected



The Czech flag contains three 2D areas, eight 1D line-segments, and six 0D points.

# Bottom-up Theory of Space

- In the bottom-up view, all spatial regions are "made of" points.
- A point has no size, hence lines, areas, and volumes have to contain infinitely (in fact non-denumerably) many of them. (Does this actually make sense?)
- On this view, the Czech flag contains infinitely many points, infinitely many lines, and infinitely many areas.
- But most of them are invisible . . .

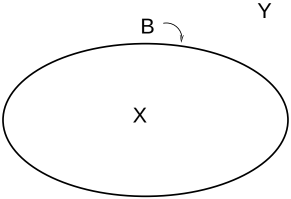
# Co-dimension

- If an $n$-dimensional region is considered as located in an $m$-dimensional space then it is said to have **co-dimension** $m - n$.
- Example: In three-dimensional space,
  - the interior of the earth has co-dimension 0;
  - the surface of the earth has co-dimension 1;
  - the coastline of Great Britain has co-dimension 2;
  - the meeting-point of Germany, France, and Switzerland has co-dimension 3.
- But if the surface of the earth is taken as the reference space,
  - the interior of Great Britain has co-dimension 0;
  - the coastline of Great Britain has co-dimension 1;
  - the meeting-point of Germany, France, and Switzerland has co-dimension 2.

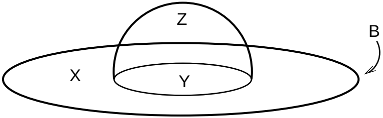
# Spatial relations

The possible spatial relations between regions depends on their dimension and co-dimension.

1. If two regions have co-dimension zero, and have the same boundary, then they are either identical or disjoint. (X and Y are the only regions with boundary B.)




2. But if they have positive co-dimension this is not necessarily the case. (X∪Y and X∪Z both have boundary B.)




In what follows, we shall assume regions of co-dimension 0.

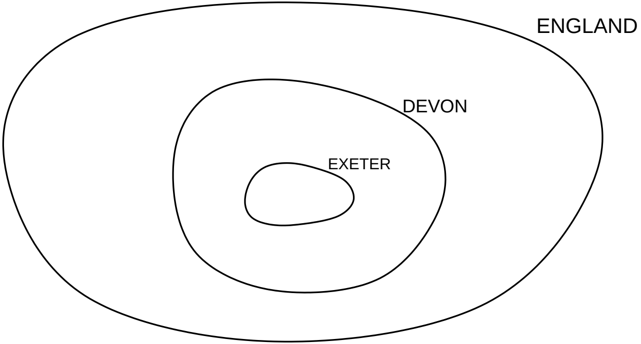
# Mereotopology

*Exeter is in Devon.*

*Devon is in England.*

What follows?

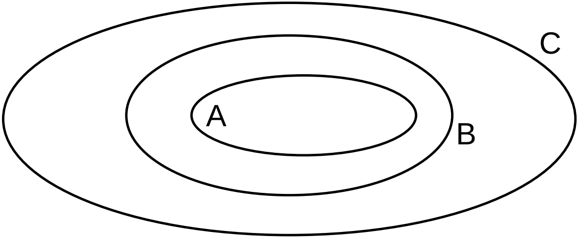
*Exeter is in England*



# A spatial reasoning rule

The Exeter–Devon–England example illustrates a general rule of spatial reasoning:

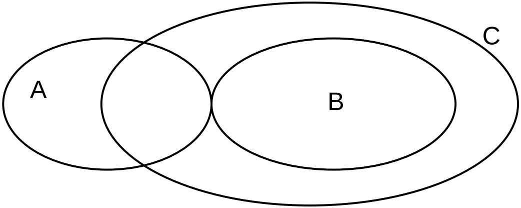
If $A$ is in $B$ and $B$ is in $C$ then $A$ is in $C$



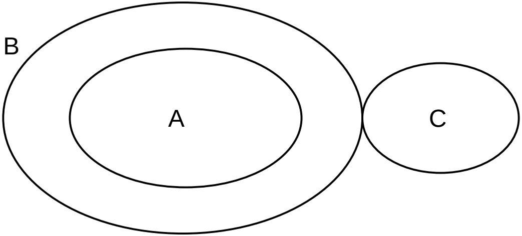

[But: The pain is in my foot; my foot is in my shoe; ...]

# Some more rules

If $A$ touches $B$ and $B$ is inside $C$ then $A$ overlaps $C$.




If $A$ is inside $B$ and $B$ touches $C$ then $A$ is disconnected from $C$.



# Spatial Relationships

The spatial terms in these inferences are

*in*
*inside*
*touches*
*overlaps*
*disconnected*

What do we mean by these?

We could characterise them **mathematically**, e.g., in terms of set theory, topology, etc.

Or we could characterise them **logically**, in terms of their relationships to each other and some agreed set of primitives.

# Parthood as a primitive

Assume that we have a primitive relation called **parthood**.

Write $\mathrm{P}(a, b)$ for '*a* is part of *b*'.

Intuitively, this means that *a* falls entirely within *b*. But we need not specify a precise meaning at this stage.

We begin by postulating two axioms for the predicate P:

- Parthood is **reflexive**:

  **PR** $\forall x \mathrm{P}(x, x)$

- Parthood is **transitive**:

  **PT** $\forall x \forall y \forall z (\mathrm{P}(x, y) \wedge \mathrm{P}(y, z) \rightarrow \mathrm{P}(x, z))$

# Defining new relations

We define the relation 'overlap' as follows

**DO** $O(a, b) =_{\text{def}} \exists x(P(x, a) \wedge P(x, b))$

i.e., '*a* overlaps *b*' means that *a* and *b* have a common part.

We can now prove that 'overlap' is **reflexive**:

**OR** $\forall x O(x, x)$

and **symmetric**:

**OS** $\forall x \forall y (O(x, y) \rightarrow O(y, x))$

# Proof of OR

For any region $a$ we have, by **PR**,

$$P(a, a)$$

and therefore

$$P(a, a) \wedge P(a, a).$$

This implies that

$$\exists x(P(x, a) \wedge P(x, a)).$$

By **DO**, this can be rewritten as $O(a, a)$. Since $a$ was an arbitrary region, we can generalise to

$$\forall x O(x, x),$$

which is **OR**.

# More relations

Given regions *a* and *b*, either they overlap or they don't.

If they don't overlap, we say they are **disjoint** or **discrete**:

$$\mathsf{DR}(a, b) =_{\mathrm{def}} \neg \mathsf{O}(a, b)$$

If they do overlap, then there are four possibilities, as follows:

1. *a* and *b* are **equal**: $\mathsf{EQ}(a, b) =_{\mathrm{def}} \mathsf{P}(a, b) \wedge \mathsf{P}(b, a)$
2. *a* is a **proper part** of *b*: $\mathsf{PP}(a, b) =_{\mathrm{def}} \mathsf{P}(a, b) \wedge \neg \mathsf{P}(b, a)$
3. *a* has *b* as a proper part: $\mathsf{PPi}(a, b) =_{\mathrm{def}} \neg \mathsf{P}(a, b) \wedge \mathsf{P}(b, a)$
4. *a* and *b* are **partially overlapping**:

$$\mathsf{PO}(a, b) =_{\mathrm{def}} \neg \mathsf{P}(a, b) \wedge \neg \mathsf{P}(b, a) \wedge \mathsf{O}(a, b)$$

# The system RCC5

The five relations DR, PO, PP, PPi, and EQ form a jointly exhaustive and pairwise disjoint (JEPD) set: any two regions must stand in exactly one of the five relations.

This system is known as **RCC5**.

RCC = Region Connection Calculus

(But also, RCC = Randell, Cui and Cohn, the authors of the paper in which it was first presented.)

But what they really mean depends on how the primitive P is interpreted.

In particular, does EQ really mean ‘equals’?

# Does EQ really mean 'equals'?

EQ($a, b$) means that $a$ and $b$ are part of each other.

Thus they exactly coincide: does this mean that they must be equal (i.e., the very same object/region/...)?

Only if the following holds:

**PA** $\forall x \forall y (\mathrm{P}(x, y) \wedge \mathrm{P}(y, x) \rightarrow x = y)$

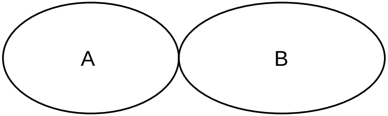
(i.e., if the 'part of' relation is **antisymmetric**).

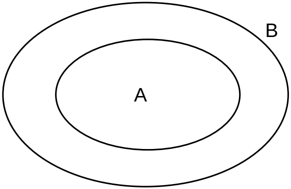
This cannot be derived from our existing axioms. We can postulate it as an extra axiom.

# 'Touches' and 'Inside'

Recall the rule:
If $A$ touches $B$ and $B$ is inside $C$ then $A$ overlaps $C$.

'$A$ touches $B$' means that $A$ and $B$ are disjoint but share part of their boundary.




'$A$ is inside $B$' means that $A$ is part of $B$ but does not share any of its boundary.

These relationships cannot be defined in RCC5!

# Introducing Connection

The RCC5 relations are called **mereological**: they belong to the theory of parts and wholes.

The relations 'touches' and 'inside' go beyond this, they are **topological**: they belong to the theory of boundaries and connection.

RCC5 can be extended to a theory called RCC8. This is a **mereotopological** theory. It is the spatial analogue of the Interval Calculus.

RCC8 is based on a single primitive relation C.

$\mathsf{C}(a, b)$ means that region $a$ is **connected** to region $b$. This means that they either overlap or touch. (Technically, the greatest lower bound for the distance between a part of $a$ and a part of $b$ is zero.)

# Axioms for Connection

The primitive relation C is stipulated to obey the following axioms:

**CR** $$\forall x \mathsf{C}(x,x)$$

**CS** $$\forall x(\mathsf{C}(x,y) \rightarrow \mathsf{C}(y,x))$$

**CE** $$\forall x \forall y(\forall z(\mathsf{C}(x,z) \leftrightarrow \mathsf{C}(y,z)) \rightarrow x = y)$$

**CE** says that connection is **extensional**: if $x$ and $y$ are connected to exactly the same things, then they are equal.

# Mereology from Topology

Parthood can be defined in terms of connection as follows:

**DP** $\mathsf{P}(a, b) =_{\text{def}} \forall x(\mathsf{C}(a, x) \rightarrow \mathsf{C}(b, x))$

I.e., *a* is part of *b* if *b* is connected to everything that *a* is connected to.

(Note: This leads to counter-intuitive results in discrete space!)

**Exercise.** Given the topological axioms **CR**, **CS**, and **CE**, and the definition **DP**, derive the mereological axioms **PR**, **PT**, and **PA**.

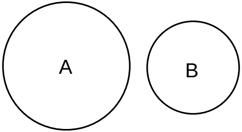
(This means that we don't need to postulate the mereological axioms if we start with connection as our primitive.)

# Mereotopological Relations

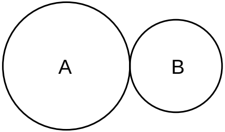
From C as a primitive, we have already defined P, PP, O, PO and EQ. We now define further relations as follows:

- $a$ is **disconnected** from $b$: $\mathsf{DC}(a,b) =_{\text{def}} \neg\mathsf{C}(a,b)$
- $a$ is **externally connected** to $b$:
  $\mathsf{EC}(a,b) =_{\text{def}} \mathsf{C}(a,b) \wedge \neg\mathsf{O}(a,b)$
- $a$ is a **tangential proper part** of $b$:
  $\mathsf{TPP}(a,b) =_{\text{def}} \mathsf{PP}(a,b) \wedge \exists x(\mathsf{EC}(x,a) \wedge \mathsf{EC}(x,b))$
- $a$ is a **non-tangential proper part** of $b$:
  $\mathsf{NTPP}(a,b) =_{\text{def}} \mathsf{PP}(a,b) \wedge \neg\mathsf{TPP}(a,b)$

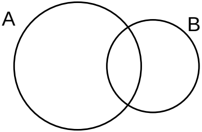
$\mathrm{DC}(a, b)$



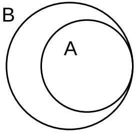

$\mathrm{EC}(a, b)$



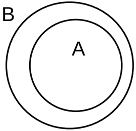

$\mathrm{PO}(a, b)$




$\mathrm{TPP}(a, b)$




$\mathrm{NTPP}(a, b)$



# The System RCC8

The inverses of TPP and NTPP are denoted TPPi and NTPPi respectively.

The eight relations DC, EC, PO, EQ, TPP, TPPi, NTPP, and NTPPi form a JEPD set known as **RCC8**.

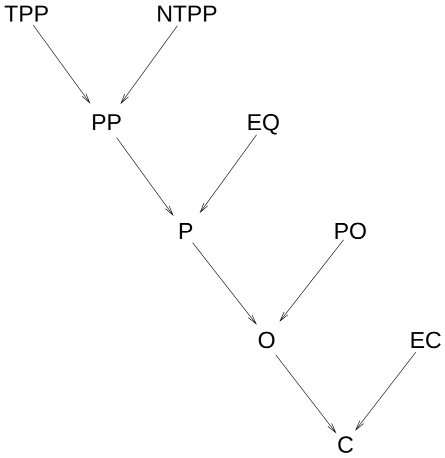
This set forms the basis for many treatments of qualitative spatial reasoning in Artificial Intelligence.

Key reference:

D. A. Randall, Z. Cui, and A. G. Cohn,
'A Spatial Logic Based on Regions and Connection',
in B. Nebel *et al.*, *Principles of Knowledge Representation and Reasoning: Proceedings of the 3rd International Conference (KR'92)*, Morgan Kaufmann (1992), pp. 165–76.

# Some Implications in RCC8

These follow immediately from the definitions.



# Reasoning with RCC8

Recall the rule:

**If $A$ touches $B$ and $B$ is inside $C$ then $A$ overlaps $C$.**

We can write this as: $\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{O}(a,c)$.
We can prove it as follows:

Assume $\mathrm{EC}(a,b)$ and $\mathrm{NTPP}(b,c)$.

$\mathrm{NTPP}(b,c)$ implies $\mathrm{P}(b,c)$, i.e., $\forall x(\mathrm{C}(x,b) \rightarrow \mathrm{C}(x,c))$. Since we have $\mathrm{C}(a,b)$ (implied by $\mathrm{EC}(a,b)$), this means that $\mathrm{C}(a,c)$.

From $\mathrm{NTPP}(b,c)$ we have $\neg\exists x(\mathrm{EC}(x,b) \wedge \mathrm{EC}(x,c))$. Since we know $\mathrm{EC}(a,b)$, this implies $\neg\mathrm{EC}(a,c)$.

We now have $\mathrm{C}(a,c) \wedge \neg\mathrm{EC}(a,c)$, and hence $\mathrm{O}(a,c)$, as required.

# Some disjunctions

These follow directly from the definitions.

- $\mathrm{DC}(a, b) \vee \mathrm{C}(a, b)$
- $\mathrm{C}(a, b) \leftrightarrow \mathrm{EC}(a, b) \vee \mathrm{O}(a, b)$
- $\mathrm{O}(a, b) \leftrightarrow \mathrm{PO}(a, b) \vee \mathrm{P}(a, b) \vee \mathrm{P}(b, a)$
- $\mathrm{P}(a, b) \leftrightarrow \mathrm{EQ}(a, b) \vee \mathrm{PP}(a, b)$
- $\mathrm{PP}(a, b) \leftrightarrow \mathrm{TPP}(a, b) \vee \mathrm{NTPP}(a, b)$

Combining them together, we infer that the RCC8 relations are exhaustive:

$$\mathrm{DC}(a, b) \vee \mathrm{EC}(a, b) \vee \mathrm{PO}(a, b) \vee \mathrm{EQ}(a, b) \vee \mathrm{TPP}(a, b) \vee \mathrm{NTPP}(a, b) \vee \mathrm{TPPi}(a, b) \vee \mathrm{NTPPi}(a, b)$$

# Refining the rule

We've proved $\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{O}(a,c)$.

Using the disjunction for O, we have

$$\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{PO}(a,c) \vee \mathrm{P}(a,c) \vee \mathrm{P}(c,a).$$

Given $\mathrm{P}(b,c)$ (from $\mathrm{NTPP}(b,c)$), if $\mathrm{P}(c,a)$ then $\mathrm{P}(b,a)$, so $\mathrm{O}(b,a)$, so $\neg\mathrm{EC}(b,a)$. But we have $\mathrm{EC}(b,a)$ (equivalent to $\mathrm{EC}(a,b)$), hence we do not have $\mathrm{P}(c,a)$.

Hence we have

$$\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{PO}(a,c) \vee (\mathrm{P}(a,c) \wedge \neg\mathrm{P}(c,a)),$$

i.e.,

$$\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{PO}(a,c) \vee \mathrm{PP}(a,c).$$

In terms of pure RCC8 relations,

$$\mathrm{EC}(a,b) \wedge \mathrm{NTPP}(b,c) \rightarrow \mathrm{PO}(a,c) \vee \mathrm{TPP}(a,c) \vee \mathrm{NTPP}(a,c).$$

This is an example of a **composition rule**.

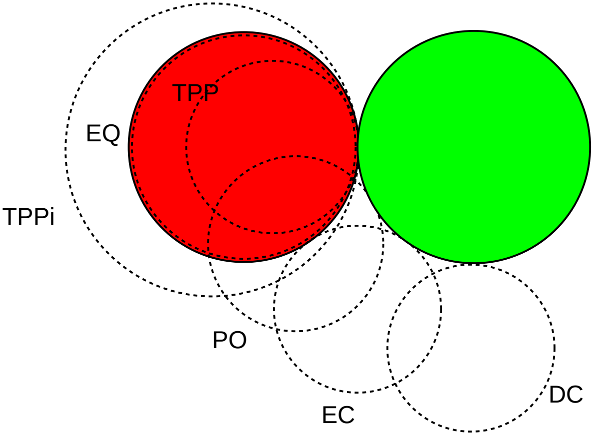
# Composition Rules

The following table gives the composition rules for all combinations of EC, PO, and TPP. The full table for all eight relations has 64 entries.

| | EC | PO | TPP |
|---|---|---|---|
| EC | DC, EC, PO, EQ<br>TPP, TPPi | DC, EC, PO<br>TPP, NTPP | EC, PO<br>TPP, NTPP |
| PO | DC, EC, PO<br>TPPi, NTPPi | *any* | PO, TPP<br>NTPP |
| TPP | DC, EC | DC, EC, PO<br>TPP, NTPP | TPP, NTPP |

# Illustration: EC with EC

Given EC($a, b$) and EC($b, c$) the diagram shows the possible relations between $a$ and $c$.
$a$ is red, $b$ is green, possibilities for $c$ have dotted outlines.



# An Example

Suppose a spatial database contains the following assertions:

*Fermanagh is a county of Northern Ireland, adjacent to the border.*
*Northern Ireland is adjacent to the Republic of Ireland.*
*Leitrim is a county in the interior of the Republic of Ireland.*
*Fermanagh is adjacent to Leitrim.*

Is the database consistent?

Translating into RCC8, we have:

$\mathrm{TPP}(f, ni)$
$\mathrm{EC}(ni, ri)$
$\mathrm{NTPP}(l, ri)$
$\mathrm{EC}(f, l)$

# Example (contd.)

1. $\mathrm{TPP}(f, ni)$
2. $\mathrm{EC}(ni, ri)$
3. $\mathrm{NTPP}(l, ri)$
4. $\mathrm{EC}(f, l)$

Using the composition table, 1 and 2 imply either

5. $\mathrm{DC}(f, ri)$

or

6. $\mathrm{EC}(f, ri)$.

We can rewrite 3 as

3′. $\mathrm{NTPPi}(ri, l)$.

Using the table again, both 5 with 3' and 6 with 3' imply

7. $\mathrm{DC}(f, l)$

This contradicts 4, so the database is **inconsistent**.

(In fact the statements are all true except 4.)

# Summary of RCC8

We have seen that RCC8 is the spatial analogue of the Interval Calculus, and that it has a composition table analogous to the composition table for the Interval Calculus.

Mathematical and computational properties of RCC have been investigated:

1. **Bennett** (1994) showed that RCC8 is decidable.
2. **Renz and Nebel** (1999) showed that the decision problem for RCC8 is NP-complete, and identified a maximal tractable fragment.
3. **Renz** (2002) gave a complete analysis of tractability for RCC8.

What about Conceptual Neighbourhood Diagram for RCC8? This is best handled in the next part when we consider combinations of space and time.